STUDIES AND THEORIES ON THE EXPANSION OF THE UNIVERSE

Pseudonym:
LAGRANGE

RESUM DEL PROJECTE

Comencem aquest projecte amb una introducció històrica de la cosmologia per entendre què significa i com ha evolucionat al llarg de la història. Així després d'introduir exemples històrics de l'interès innat dels humans pel cel i els fenòmens astronòmics, estudiem la història dels principals científics que han fet possible el model cosmològic actual. Entre aquests hi trobem n'Albert Einstein, en Georges Lemaître, n'Henrietta Levitt, n'Edwin Hubble i en George Gamow. Destaquem sobretot el paper de n’Henrietta Levitt com a dona de ciència a la cosmologia, ja que a principis i mitjans del segle XX aquest era un camp dominat exclusivament pels homes.

A continuació, revisem els conceptes principals del model cosmològic actual definint els l’homogeneïtat i l’isotropia. Des d'un punt purament clàssic, emprant la mecànica newtoniana i ajudant-nos dels conceptes definits, derivem les equacions de Friedmann sense la constant cosmològica. Per fer-ho aconseguim l'equació de l'energia total d'una galàxia allunyant-se de nosaltres a causa de l'expansió de l'univers però atreta gravitatòriament, i llavors l'homogeneitat de l'univers ens permet exportar-ho a un cas més general per tot l'univers.

Hem vist que en funció de l'energia total de la galàxia tindrem tres evolucions diferents, de manera anàloga a un coet sortint d'un camp gravitatori a una velocitat determinada. Hem començat pel cas d'energia 0, equivalent a la d'un coet sortint a velocitat terminal i hem vist que a l'infinit la velocitat és 0 i, per tant, tindríem un univers estàtic. Llavors hem vist que pel cas d'una energia positiva l'univers s'expandiria amb velocitat constant a l'infinit, finalment hem vist el cas d'energia negativa en què l'univers col·lapsa (anàleg del coet caient a la terra).

Comparant les equacions de Friedman derivades amb les equacions de camp de la relativitat general a partir de la mètrica FLRW hem vist que el valor de l'energia total està relacionat amb la curvatura de l'univers, essent l'energia positiva el cas d'un univers hiperbòlic, energia 0 per un univers estàtic i energia negativa per un univers esfèric.

Hem agafat el cas d'energia 0 i aïllat la densitat de l'equació de Friedman per obtenir una expressió de la densitat de l'univers pla en funció del valor de la constant de Hubble. Finalment, hem vist que un univers dominat per radiació (com és el cas de l'univers primigeni) hauria tingut un ritme d'expansió més elevat al principi.

Hem vist els conceptes principals de la relativitat especial. Hem procedit a derivar les transformacions de Lorentz i ho hem visualitzat amb l'ajuda de diagrames d'espai-temps. Finalment, hem introduït la mètrica de l'espai-temps amb el temps propi com a valor absolut igual per a tots els observadors.

Hem donat algunes pinzellades a la relativitat general i hem introduït les equacions de Friedman amb la constant cosmològica, la qual permet introduir l'acceleració de l'expansió de l'univers causada per l'energia fosca.

En la part experimental hem calculat la distància de la supernova SN 2022 hrs de tipus I-A. Amb el telescopi de l'Associació Astronòmica de Sabadell, mitjançant fotometria (a partir de la seva brillantor), hem verificat que encaixa en la corba de brillantor de la supernova proveïda per Yasuo Sano de l'observatori de Nayoro. A partir del valor màxim de brillantor d'aquesta corba, donat que és una supernova de tipus I-A, hem pogut calcular la seva distància. Hem calculat la velocitat de recessió a partir del seu espectre (publicat a la base de dades WISeREP). A partir de la distància i la seva velocitat hem aconseguit obtenir una proporció orresponent a la constant de Hubble.

A partir del valor de la constant, suposant una expansió líneal, hem calculat l'edat de l'univers, que ha resultat en uns 16,67 milers de milions d'anys. Seguidament, hem calculat la seva densitat amb l'equació derivada anteriorment a partir de l'univers amb energia 0 / pla. Hem obtingut un valor de $6,47 \cdot 10^{-27}$ kilograms per metre cúbic. Hem estimat que aquesta densitat és l'equivalent a la d'un gra d'arròs de 0,02 grams en el volum de 119 planetes Terra o 2 planetes Neptú.

Per acabar s'ha fet un annex on s'utilitza la mètrica de l'espai-temps del temps propi per tenir un lagrangià absolut, amb el mateix valor per tots els observadors a partir del qual derivar l'energia total d'un sistema relativista i trobar la relació entre energia i massa coneguda amb la famosa equació $E = mc^2$.

"It took less than an hour to make atoms, a few hundred millions years to make stars and planets, but it took five billion years to make man"

GEORGE GAMOV

ACKNOWLEDGEMENTS

First of all, I would like to thank my research supervisor, [redacted], for his advice and willingness to always help me.

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Finally, I would like to express my special thanks to my parents for their patience and for being there when my strength was failing.

ABSTRACT

We start this project with an historical introduction of cosmology to understand what it means and how it has evolved throughout history. Then we see some of the main concepts of how the universe expands and apply them to the Newtonian classical mechanics laws to end up getting the true Friedman equations without the cosmological constant. After that we review some of the main concepts of the theories of relativity and we derive the basics of special relativity. We move on to the practical part to get a value for the Friedman equations. We end up relying on the results from astronomical databases. We get a value close to the estimated one by approximately 15% of its value. At the end, we get the values for the age of the universe and its density applying what we have learned from the Newtonian Friedman equations, assuming that our universe is flat and it has had a linear expansion.

ABSTRACT (CAT)

Comencem aquest projecte amb una introducció històrica de la cosmologia per entendre què significa i com ha evolucionat al llarg de la història. A continuació, veiem alguns dels conceptes principals de com s'expandeix l'univers i els apliquem a les lleis de la mecànica clàssica newtoniana per acabar obtenint les veritables equacions de Friedman sense la constant cosmològica. Després repassem alguns dels conceptes principals de les teories de la relativitat i obtenim els fonaments de la relativitat especial. Passem a la part pràctica per aconseguir un valor per a les equacions de Friedman. Acabem confiant en els resultats de les bases de dades astronòmiques. Obtenim un valor proper a l'estimat en aproximadament un 15% del seu valor. Al final, obtenim els valors de l'edat de l'univers i la seva densitat aplicant el que hem après de les equacions newtonianes de Friedman, suposant que el nostre univers és pla i ha tingut una expansió lineal.

CONTENTS

1. INTRODUCTION (FR)

Comme nous allons le voir dans notre recherche historique, l'univers a été une chose qui a intéressé l'être humain. Moi-même je m'y suis intéressé à partir du moment où j'ai pu regarder et contempler le ciel nocturne.

Pendant l'histoire on a essayé de le connaître à travers des mythes. Aujourd'hui on a deux outils très puissants, la méthode empirique et la logique et les mathématiques. C'est avec ces outils qu'on va faire un voyage qui commence avec l'histoire de la cosmologie et qui finisse avec des résultats concernant notre univers. C'est sûr que lorsque le lecteur lit le fait que nous voulons calculer des choses comme l'âge de l'univers, il peut penser que le travail va être compliqué et il pourrait se déconnecter, surtout quand les équations commencent à apparaître. Cependant, j'encourage le lecteur à essayer de ne pas le faire, car il n'y a que des mathématiques de base nécessaire pour comprendre les concepts.

Ironiquement, calculer des choses comme l'expansion de l'univers ou son âge, c'est facile, mais cela demande un gros budget. C'est pour cette raison que nous nous limiterons à faire des observations qui peuvent vérifier que les données de différents observatoires utilisées sont correctes, en primarisant nos résultats chaque fois que ce soit possible.

En dépit de la complexité théorique et la limitation de budget, on va connaître les caractéristiques de notre univers sans utiliser ni des mathématiques très compliquées (la partie la plus compliquée sera dans l'annexe et elle ne sera pas nécessaire pour comprendre le travail) ni un budget trop élevé.

Malgré tout, cela on va découvrir qu'est-ce que c'est l'univers, on va calculer son rythme d'expansion et connaître ses propriétés principales. Donné que c'est une zone très complexe et qui a déjà été étudiée, c'est très difficile de faire une recherche marquante pour le monde scientifique. En dépit de ça on peut encore y contribuer avec des résultats qui puissent vérifier le modèle cosmologique actuel et on peut y donner un point de vue plus divulgatif, en rapprochant la cosmologie et la physique théorique à des personnes ayant une formation académique de base. Dans l'annexe on va faire un petit peu plus de recherche théorique pour répondre à des questions plus compliquées, mais il ne sera pas nécessaire pour comprendre la recherche basique faite.

2. KEYWORDS

- **Astronomy**:

  It is the science that studies the celestial bodies of the universe: stars, planets, natural satellites, asteroids, comets and meteoroids, interstellar matter, nebulae, dark matter, galaxies, etc. It also studies astronomical phenomena: supernovae, quasars, pulsars, cosmic background radiation, black holes, etc. And the laws that govern it. It is related to physics through astrophysics, chemistry to astrochemistry, and biology to astrobiology.

- **Cosmology**:

  It is a branch of astrophysics dedicated to studying the dynamics and relationships between the different bodies that make up the universe, tries to theorize about the origin and large-scale evolution of the universe.

- **Newtonian Cosmology**:

  The synthesis of the world system, formulated by Isaac Newton in his Mathematical Principles of Natural Philosophy (1687), where he exposes his laws of motion and the force of the universal attraction of bodies in direct ratio of their mass and inverse ratio of the square of its distance. The universe is governed uniformly at all its points by the same laws, the same forces of inertia and gravity.

- **Relativity**:

  In physics, the term relativity is used to refer to mathematical transformations that can be applied to describe phenomena in different reference systems. Described by Albert Einstein, it predicts phenomena such as the different perception of time according to the observer and the deformation of the space-time fabric.

- **Special Relativity Theory**:

  Published by Albert Einstein, described motion in the absence of gravitational fields. They will deduce that, according to Maxvell's equations, electromagnetism did not follow Newton's laws when the observer's reference changes when it is a physical problem from the point of view of others.

- **General Relativity Theory**:

  Published by Albert Einstein, described gravitation in physics, unifying Special Relativity with Newton's laws. He described gravity as a geometric property of space that described the severe curvature related to energy and movement.

- **Spectroscopy**:

  It is dedicated to the study of light that is absorbed or emitted by an object. It breaks down light and measures the different wavelengths of visible and non-visible light. It is used in astrophysics to determine the nature and physical properties of distant stars and galaxies, such as their chemical composition and movement, by means of the Doppler effect.

- **Doopler effect**:

  It is the change in frequency of a wave produced by the relative movement of the source with respect to the observer. When a light-emitting object approaches the observer, the light waves increase in frequency and move towards the blue colour of the spectrum, if it moves away, the waves move towards the red.

3. THEORETICAL RESEARCH

3.1. History of Cosmology

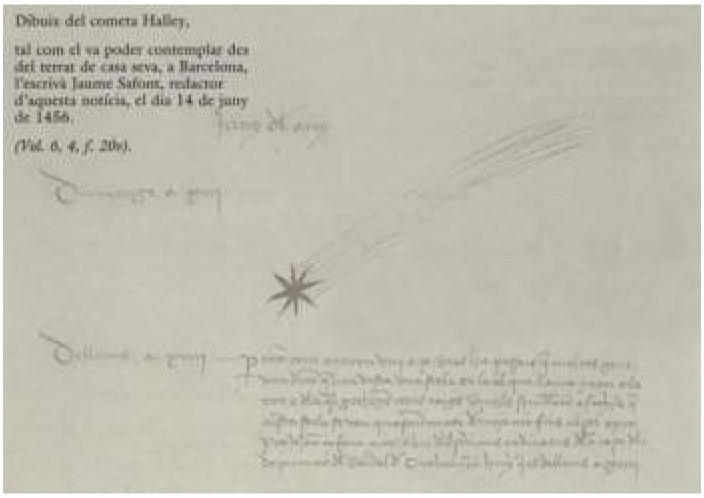
3.1.1. interest in the universe

Our investigation on the history of human research of the universe could start in the ancient ages. The Greeks weren't probably the first ones to think about the universe, since in the bronze ages people were already asking themselves about the universe, and were likely to predict astronomical events such as solar eclipses; that's the case of some ancient American tribes such as the Puebloans[1].

Already in ancient times the research of the universe wasn't just motivated by some necessities such as having a calendar for agricultural purposes as curiosity is part of human nature. We all ask ourselves at some point about our existence and its meaning and we try to find it in the immensity of the cosmos from which we come. That's why even though the research of the universe hasn't had such a technological importance throughout history, we've kept trying to answer existential questions by looking at the sky. Even in the middle ages, when everything was explained by the existence of God, people continued asking themselves questions about astronomical phenomena which were seen as supernatural and impressed everybody. That's the case of Halley's comet when it was seen in 1456, the passage of the comet is described in some texts of the time, such as the "*Dietari de la Generalitat*"[2] [3]*:*

"Dilluns, a XIIII. Per tant com, entorn VIII o X dies ha passats, que moltes gents van dient que han vista una stela en lo cel qui llança gran claror e de la qual procehexen certs raigs vermells semblants a foch, e que aquesta stela se veu quascun matí de mijanit fins al sol exit, per ço, jo, Jacme Çafont, notari e hu dels scrivans ordinaris de la casa de la Deputació del General de Cathalunya, huy que és dilluns, a XIIII de juny M CCCC LVI, volent veure si és ver ço que·s diu de aquesta stela, me són levat entre II e III hores passada mijanit e són muntat alt, en la torra de casa mia. E, de fet, é vista una stela entre grech e tremuntana, de la qual procehien grans raigs de claror qui partien de la dita stela e signaven entre llebeig e migjorn, e podien haver de larcha a bon arbitre de XVIII en XX palms, e d'ample o de gros un bon palm, la qual stela e raigs eran fets en la manera dessús designada. Déus vulla que bon senyal sie, que los hòmens de la buscha, qui concorren en aquesta temporada, han tal adobada aquesta ciutat que ab poques males ventures hauríem prou sobre ço que ja havem."

Translation: *"Monday 14th. Given that for 8 or 10 days many people have been saying that they have seen a very bright star in the sky and from which red light rays come out like if they were flames, and that this star is visible every night from midnight until sunrise, that's why, I, Jacme Çafont, notary and one of the regular scribans of the house of Deputació del General de Cathalunya, today Monday 14th june of 1456, wanting to know if what is said about this star is true, I woke up between 2 and 3 hours after midnight and I have climbed to the top of my home's tower. And, indeed, I have seen a star between north and northeast, from which big bright light rays came from and reached to between the south and southwest, and they could be between 18 and 20 hands long and one hand wide as shown in the drawing below. God willing, it's a good sign, because the merchants that are around this time have ill-treated the city so much that with a few bad news we would have enough from what we already have."*



Picture representing Halley's comet drawn on June 14th 1456.
Volume I. P. 514. (1411-1539).

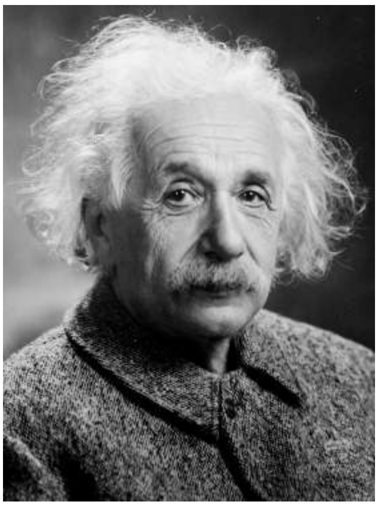
It is clear that humans have always been curious about the universe due to our innate curiosity for the cosmos, and that's in all likelihood the main motivation of modern cosmology.

3.1.2. Beginnings of cosmology

As a branch of physics, cosmology is relatively new. For the purpose of this research we will only discuss the history of cosmology as the study of the universe on a large scale, which began in the last century after the emergence of general relativity, although most of the cosmological discoveries of the time can be explained with Newtonian mechanics.

We can say that cosmology began in 1915 after Einstein published its theory of general relativity. To understand concepts such as the curvature of the universe or its expansion, we have to understand how special and general relativity work and how they changed our way of thinking about the universe.

By the end of the 19th century most of the physics of our ordinary world had been developed and it seemed that soon physics could explain everything. However, there were still some problems that classical mechanics could not solve so we can now talk about two main problems, that seemed unimportant but they led to two revolutions in physics that changed the way we see the world, one of them was quantum mechanics and the other was the theory of special and general relativity.
Special relativity showed that the perception of time is not an absolute thing, but changes depending on the speed and position of the observer, this causes a phenomenon called time dilation. Einstein went even further and showed that there is no such thing as absolute simultaneity between actions, since the order of events can also change.
On the other hand, general relativity is a theory of gravity that defines the concept of space-time fabric and explains gravitational fields as a deformation of it. It is a very hard theory that uses advanced mathematical concepts like tensors. As they are complex theories and need to be explained properly you can read more about the theory of special in the appendix.



“I am convinced that He (God) does not play dice with the universe.” - Albert Einstein

Orren Jack Turner, Princeton, N.J

The most important part of general relativity for cosmologists is the cosmological constant. While developing the field equations of general relativity, Einstein found that his own equations about the behavior of space-time described an expanding universe, which was something unthinkable at the time. Einstein surprisingly always followed his common sense and believed in a statical universe, he analyzed his equations and found that if he added a constant with a specific value to his field equations he would have a static universe. That's how the concept of cosmological constant was born, it's a value that describes the expansion of the universe. Soon as we will see, other scientists proved that the universe is indeed expanding to the point that Einstein would regret his error and would call it the worst blunder of his life[4].
However, nowadays it's thought that the expansion of the universe isn't constant but it's accelerated, that's why the cosmological constant appears again in general relativity.

The first one to challenge Einstein's point of view of a static universe was the Belgian priest and astronomer Georges-Henri Lemaître. He was fighting in the First World War when he read a book on cosmology written by Jules-Henri Poncaré. After the war, in the 1920's he enrolled at Cambridge University and later at Harvard University to study astronomy. He was the first one to come up with the idea of the big bang, however it was a time of revolutions in cosmology and other astronomers, such as Hubble or Slipher who were already accumulating data about galactic redshifts—as we will do in this project—were about to arrive at the same conclusion from an experimental basis.

"There is no conflict between science and religion." - Georges Lemaître

Image Source: physicstoday.scitation.org

It would be between 1927 and 1933 that Lemaître made the first version of Big Bang theory, which he called "*Hypotèse de l'atome primitif*". During that time there was a change in quantum mechanics that led to new ways of seeing our world, that's why he described the origin of the universe as a single atom of energy which would have begun dividing at a very fast rate from which space and time would unfold.

Lemaître showed his theory of the primordial atom to Einstein who answered him: "*Your calculations are correct, but your physical insight is abominable.*" Even his own teacher Arthur Eddington rejected his idea: "*It has seemed to me that the most satisfactory theory would be one which made the beginning not too unaesthetically abrupt.*" Even though his calculations were right, his theory was rejected because it was against common sense. He wrote a letter about his theory which was published in Nature recalling what his former teacher told him:

"SIR ARTHUR EDDINGTON states that, philosophically, the notion of a beginning of the present order of Nature is repugnant to him. I would rather be inclined to think that the present state of quantum theory suggests a beginning of the world very different from the present order of Nature. Thermodynamical principles from the point of view of quantum theory may be stated as follows : (1) Energy of constant total amount is distributed in discrete quanta. (2) The number of distinct quanta is ever increasing. If we go back in the course of time we must find fewer and fewer quanta, until we find all the energy of the universe packed in a few or even in a unique quantum.

Now, in atomic processes, the notions of space and time are no more than statistical notions; they fade out when applied to individual phenomena involving but a small number of quanta. If the world has begun with a single quantum, the notions of space and time would altogether fail to have any meaning at the beginning; they would only begin to have a sensible meaning when the original quantum had been divided into a sufficient number of quanta. If this suggestion is correct, the beginning of the world happened a little before the beginning of space and time. I think that such a beginning of the world is far enough from the present order of Nature to be not at all repugnant." [5]

Soon Lemaître became a celebrity appearing in newspapers such as the New York Times. Already in 1931 Einstein had realized Leimaître was right, he then referred to his theory as "the most pleasant, beautiful and satisfying interpretation." Later his old teacher Eddington, who at first had dismissed his theory, would publish it in the

Royal Astronomical Society. That's how his first big bang theory became widely known.[6]

Picture of Einstein and Lemaître in California in 1933.

Foto: Anonymous/ AP

3.1.3. Relevant cosmologists

Henrietta Swan Leavitt was an American astronomer, she was a deaf woman who worked at Harvard College Observatory studying Cepheid variable stars. It is a type of star that pulsates, varying in diameter and temperature producing a change in its brightness following a period. Leavitt found a relationship between the period and its brightness, hence creating a new way to measure the distance of galaxies by measuring the period and brightness of stars within the galaxy.

Picture of Henrietta Leavitt working at a desk in Harvard College Observatory.

Source: Harvard College Observatory

That was the basis of Hubble's work, who used Leavitt's technique to calculate the distance to Andromeda measuring the periods and apparent brightness of its Cepheid stars. Thanks to Leavitt's work, Hubble discovered that the universe was indeed much bigger than it was thought to be.

Edwin Hubble was an American astronomer. He graduated in law from the University of Oxford and after a short period practicing as a lawyer, he went on to study astrophysics at the University of Chicago where he received his doctorate in 1917. After fighting in World War I he began to work at the Mount Wilson observatory where he had access to the Hooker telescope, the largest of its time (2.54 m diameter). He stayed at Mount Wilson for the rest of his life.

His contribution to astronomy was so crucial that many consider him the father of cosmology.

According to the studies of Melvin Slipher who discovered that there were nebulae moving away from the sun (its spectrum clearly deviated towards the red color), he decided to deepen the subject and thanks to his mastery of photography he obtained two plates of M31 where stars could be seen inside what, until now, had been considered a nebula (set of dust and gases) and which he clearly identified as an object external to the Milky Way, another galaxy, Andromeda. This is how they discovered that the universe was made of galaxies. Through spectroscopy they saw that most were moving away from Earth (they had a red-shifted spectrum) and the more distant they were the faster they moved, making it possible to state that the speed of recession is proportional to its distance. Lemaître had reached this same conclusion, albeit theoretically, this is why it is called the Hubble-Lamaître Law. This allows us to affirm that, as Einstein had predicted, the universe is expanding.

Hubble determined that there is a constant that relates distance to speed, a constant that takes his name, thanks to his research they were able to calculate the age of the universe at 15,000 million years. Once the discovery was verified, he dedicated himself to classify galaxies according to their shape and to create the Hubble Atlas of galaxies, which was the result of 30 years of observations and which was published after his death.

“The history of astronomy is a history of receding horizons.” - Edwin Hubble

Johan Hagemeyer, 1931. Huntington Digital Library

George Gamow was an American physicist of Russian (Ukrainian) origin known for his work in biochemistry and astrophysics. Born in Odessa, he studied at the University of Novorossiya and later at the University of Leningrad where he obtained a degree and doctorate. He completed his studies in Göttingen (Copenhagen) with Niels Bohr and in Cambridge with Ernest Rutherford. Afterwards, he became a University of Leningrad professor and followings a short stay at the Pierre Curie Institute in Paris and the University of London, he was hired by George Washington University in the United States, where he stayed until 1956. During the Second World War he was called by the government to be part of the group of scientists to work on the atomic bomb project.

His most outstanding studies were done with Ralph Alpher in the development of the Big Bang theory (explosion of a primordial atom of high density from which all the chemical elements came out), the theory proposed by Lemaître and that Gamow and Alpher contributed to its deepening and dissemination. Primordial nucleosis has been demonstrated by measurements of the expansion of the universe. An article, published by his students, Alpher and Herman, predicted the existence of a background radiation coming from the big initial explosion that was later discovered by accident. It is called background radiation because it’s found everywhere in the universe and according to their theory it would fill the entire cosmos, therefore it should be the same in any direction.

He published articles on the formation of the Solar System.
He developed the Gamow-Teller theory on the internal structures of red giant stars, delved into the stellar energy cycle and was among the first to deny the cooling of the Sun, predicting instead its heating and expansion what would cause the extinction of life on Earth.

“It took less than an hour to make the atoms, a few hundred million years to make the stars and planets, but five billion years to make man!” - George Gamow

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Work Cited

1. Malville, J. M. K. (2008). *A guide to prehistoric astronomy in the Southwest*. Johnson Books
2. *Dietari General de Catalunya .* Volume I. P. 131. (1411-1539).
3. For further information about astronomical references in Dietari de la Generalitat see: JOSÉ, M., & MARCO, F. J. Astronomical references in the first volume of the Dietaris de la Generalitat de Catalunya (1411-1539) and the the Dietari o llibre de Jornades de Jaume Safont (1411-1484).
4. Gamow, G. (1970). *My Worldline*. Viking Press. p44
5. LEMAÎTRE, G. The Beginning of the World from the Point of View of Quantum Theory. *Nature* 127, 706 (1931). https://doi.org/10.1038/127706b0
6. Smoot, G., & Davidson, K. (2007). Chapter 3. In *Wrinkles in time: Witness to the birth of the universe* (pp. 53–57). essay, Harper Perennial.

3.2. Newtonian Cosmology

In this part of the project I will derive Friedman's equations and theorize the expansion of the universe based on Leonard Susskind's lectures on Cosmology and a course offered by "*L'Astronòmica de Sabadell*". From this we will have enough theoretical knowledge to calculate things such as the density or even the curvature of the universe from Hubble's constant.

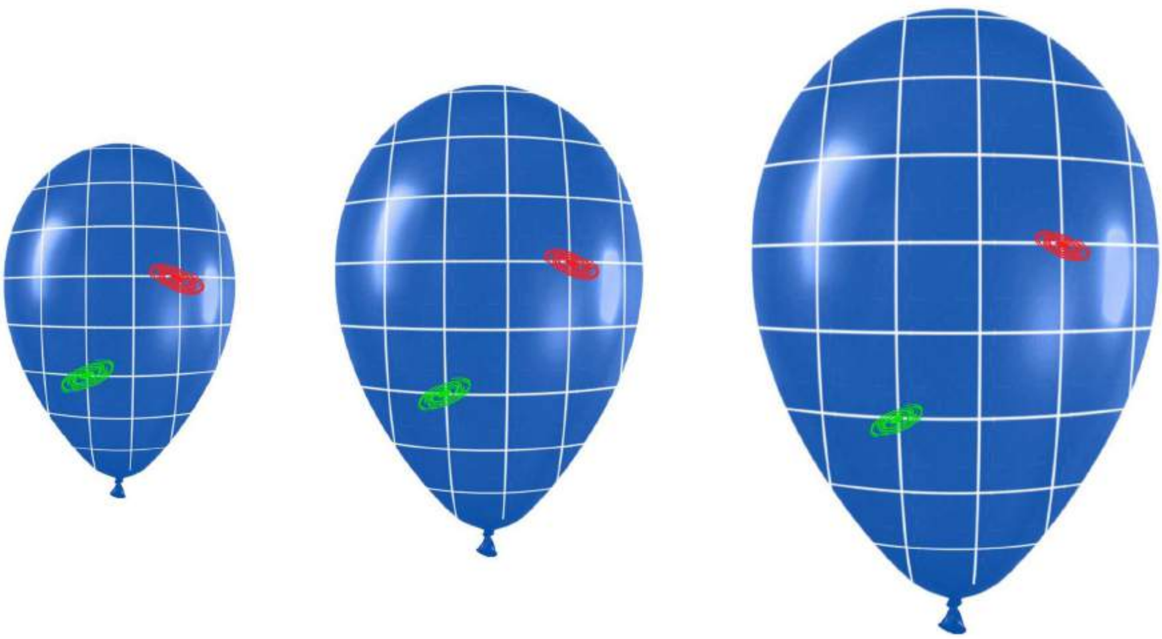
3.2.1. The expansion with Newtonian mechanics

The expansion of the universe, even though it's a property of space time that expands, can be explained with Newtonian mechanics. That's because galaxies can be treated as simple objects which move at non relativistic (slow) speeds and have a gravitational attraction that works in a similar way as that of stars or planets since in most cases there aren't any huge gravitational interactions. Consequently we can see galaxies as simple dots in space that are subject to some velocity of expansion and to Newtonian laws of gravity.

In order to do that we have to review some of the basic principles of the expanding universe. To start with we have to see our universe as homogeneous and isotropic. An isotropic universe means that no matter what direction you look at, you will see the same. An homogeneous universe means that it looks the same at all locations. One could say that one thing is a consequence of the other, even though homogeneous systems are often isotropic too, in some cases there could be an homogeneity without isotropy, or isotropy without homogeneity. For example we could look at a wall with bricks, it does look the same in all directions, but from a brick you won't see the same thing in all directions, since in the right you may see the left part of a brick, in the upper left a corner, etc. But our universe is both isotropic and homogeneous, that is to say that on the whole it looks pretty much the same and you will see the same at any location. One could object by saying that the universe isn't neither isotropic nor homogeneous since it doesn't look like it locally, if you look in different directions from earth you will see different things. But we have to have in mind that there are 10^{11} galaxies in the observable universe with 10^{11} stars each and it's in big scales that it looks homogeneous and isotropic.

That's how the cosmological principle is introduced, in the past it was thought that the earth was the center of the universe and we were at a privileged place. This principle teaches us that there's no special point in the universe, thus there's no thing such as the center of the universe. This would be essential to understand how our universe expands.

As it has been shown before, the expansion of the universe is a fact. Following the cosmological principle we know that it expands the same way in all directions at all locations. One might think that the balloon has a center from which everything expands, but that would be wrong, we can imagine the expanding universe as a balloon being inflated, its surface being the universe. In our analogy the space would be bidimensional and there wouldn't be such a thing as a center. We could also see it as chess grids with each one expanding uniformly, there wouldn't be such a thing as a center but each greed would expand.



Balloon that represents how the universe expands, as we can see in this analogy every grid expands causing the galaxies to move away from each other.

3.2.2. Derivation of Friedman equation.

Let's call the number of greeds between two points Δx. The two galaxies drawn will always remain in the same point in the grid, the grids are what will expand. Let's suppose there's a scale parameter that varies in function of the universe's expansion, when grids are small the parameter will be small, when grids increase in size the parameter will increase. We will call the scale parameter a, since we expect the universe and hence the grids to grow in time the scale parameter may be time dependent. It's easy to see that the distance between grids will be given by the number of grids multiplied by the scale parameter:

$$D = a(t) \cdot \Delta x$$

It's obvious that the velocity between two galaxies due to the expansion will depend only on "a", since the number of grids will be the same, in the expansion of our universe the only thing that will change is the size of the grids, hence the scale parameter:

$$v = \dot{a}(t) \cdot \Delta x$$

Where:

$$\dot{a}(t) = \frac{\mathrm{da}}{\mathrm{d}x}$$

Now we can divide the velocity by the distance, note that the Δx will cancel. We will obtain a value that tells us how the universe expands. Given a distance we will be able to know the velocity of expansion thanks to the value:

$$\frac{v}{D} = \frac{\dot{a}(t)}{a(t)} = H(t)$$

That's the Hubble constant, it relates the velocity of remote galaxies with their distance. Note that "H" depends on time, thus it's not really a constant, we call it a constant because it's the same throughout all space, since it doesn't depend on "Δx". But it can change with time if the expansion rate changes, consequently it isn't really a constant. From that we can write Hubble's law:

$$v = H \cdot D$$

Note that for now we have referred to the distance of grids between two galaxies as "Δx", but in fact we are in a three dimensional space, therefore the distance between two galaxies in terms of grids will be given by:

$$R = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}.$$

Now since the universe is homogeneous and isotropic we will define the amount of mass in a given grid as ν. Consequently the mass in a big enough region will be:

$$M = \nu \cdot \Delta x \ \Delta y \ \Delta z$$

We will also define the volume of a given region as:

$$V = a^3 \cdot \Delta x \ \Delta y \ \Delta z$$

Finally we will define the density of a given region as the mass divided by the volume. It's easy to see that the $\Delta x \, \Delta y \ \Delta z$ will cancel. Consequently the density of a given region will be:

$$\rho = \frac{\nu}{a^3}$$

Note that since the universe is homogeneous at large scales the density won't depend on the location, that's why we can't see any term related to position in the last equation.

One could mistake density for the amount of mass in a given grid "ν", but they are not the same. The amount of mass in a given grid will always remain the same, no matter if the grids expand, the mass in a grid will always be the same. On the other hand the density depends on the volume, consequently it will drop if the universe expands since the mass will remain the same but the total volume of a given grid and the universe itself will increase.

Now let's study how a galaxy in a given grid will move relatively to us. We will center ourselves in the center of coordinates, in fact it doesn't matter where we chose the origin to be, we will see that later.

If we suppose the mass in our universe is uniformly distributed then we just take in account the mass in a circle whose radius is the distance between us and the galaxy. That's a proven theorem, it's called the shell theorem. Newton found that a sphere can be built from an infinite number of small rings, in fact that's why he did his Principia mathematica. He thought that if he could prove that in a ring where the mass was uniformly distributed, the gravitational force was always pointing at the center he could prove the same for a shell and consequently for a uniform sphere. For now we will have to believe it.

From what we have said before, the gravitational force over the galaxy will be:

$$F = \frac{-GMm}{D^2}$$

Where "G" is the gravitational constant, "m" is the mass of the galaxy, "D" is the distance between the origin (us) and the galaxy, and "M" is the mass inside the circle of radius "D".

If we divide "F" by the galaxy's mass we will have an expression for the acceleration of the galaxy, which we know it's equal to the double time derivative of the distance. Since we know that the time derivative of the velocity is:

$$v = \dot{a}(t) \cdot R$$

Remember that since we are in a three dimensional space we changed “Δx” by “R”. It’s easy to see that the number of grids will never change with time from what has been explained before, consequently the acceleration will only depend on a, hence:

$$A = \ddot{a}(t) \cdot R$$

Now from the acceleration given by the gravitational force we get:

$$A = \frac{-MG}{D^2} = \ddot{a}(t) \cdot R$$

Now we can substitute “D^2” by “$a^2 \cdot R^2$”:

$$A = \frac{-MG}{a^2 R^2} = \ddot{a} \cdot R$$

If we rearrange the equation and divide both sides by a we get:

$$\frac{-MG}{a^3 R^3} = \frac{\ddot{a}}{a}$$

Notice that as we said earlier since the mass is distributed uniformly we just have to take into account the volume of a sphere of radius “R”. The volume of the sphere will be:

$$V = \frac{4}{3}\pi r^3$$

Where the radius is the distance between the center and the galaxy. Hence we can substitute the radius with “aR”:

$$V = \frac{4}{3}\pi a^3 R^3$$

Notice that if we multiply the denominator of the equation we have by "$\frac{4}{3}\pi$" we will get the expression for the volume. We will multiply both sides by it:

$$\frac{\ddot{a}}{a} = \frac{-\frac{4}{3}\pi MG}{\frac{4}{3}\pi a^3 R^3} = \frac{-\frac{4}{3}\pi MG}{V}$$

Now we have an expression where we have a mass divided by a volume, that's the density:

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G\rho$$

Since the universe is homogeneous at big scales the density of the universe is the same everywhere. We have obtained an equation that tells us how the universe expands, we have a ratio that tells us how the expansion accelerates from a given value of "a". The important thing is that the equation no longer depends on the observer since the "R" has disappeared, no matter where you are the acceleration of the expansion will be given by this equation. Notice another thing, in order to have a static universe the ratio "$\frac{\ddot{a}}{a}$" should be "0", since in a static universe "a" would have a constant value different from "0" that wouldn't change in time, thus the double time derivative would be "0". But that's not possible with this equation, neither "π" not "G" can be "0" since they are constants and "ρ" (the density) must be different from "0" as there's mass in our universe. One could say that the minus sign could have some meaning since it could be related to whether the universe is expanding or contracting, but it just tells us that the acceleration is pointing to the center (earth), it doesn't tell us whether the velocity is towards the earth or not. The important thing is that, from this equation we can tell that our universe must be either expanding or contracting and it doesn't depend on where you are, it will be the same at any location.

Taking into account that cosmology as a branch of physics appeared in the last century, one could ask how someone like Newton didn't theorize the expansion of the universe since the basics of cosmology can be explained with Newtonian physics. It's true that Newton himself could have given a theoretical basis to develop these equations. However we have to think that an expanding universe was something that was against common sense, as we have explained before, Einstein himself in the XXth thought it was impossible for the universe to be expanding, in the XVIIth century nobody could think about an expanding universe or a uniform universe, at that time the universe was perceived as just the solar system and the cosmological principle was still in debate, scientists were just starting to abandon the idea of a geocentric universe.

Now remember that we can define the density in terms of “a”, consequently we get:

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi\frac{G\nu}{a^3}$$

That’s an equation we could try to solve, but note that it’s a second order differential equation.

3.2.3. Energies of the universe, escape velocity

There’s a simpler version of the equation using energies. Think about the expanding universe, we can say from the last equation that the expansion is slowing down because of gravity. We can take the analogous of the escape velocity. The escape velocity tells us whether a body will ever escape from a gravitational field or will fall back, in this case it will tell us whether the universe will expand forever or it will slow down enough to implode. Let’s think about energies, we will have some kind of kinetic energy due to the velocity of a galaxy in the expanding universe, the potential energy will be given by the gravity that makes the acceleration slow down. As in a body escaping from gravity if the total energy is “0” then we will have a universe that will slow down to “0” in the infinity, if the total energy is negative as a body would fall back the universe would collapse, if it’s positive the universe will expand forever.

So let’s take the example of a galaxy in a coordinate system where we are at the center, the total energy will be given by:

$$E = \frac{1}{2}mv^2 - \frac{GMm}{x}$$

The first term being the kinetic energy and the second one the potential energy due to gravity. “x” stands for the distance between us and the galaxy, the small “m” stands for the mass of the galaxy and the big “M” stands for the mass in the sphere of radius x. As we have defined it before, x and v will depend on a since:

$$x = a(t) \cdot R$$

$$v = \dot{a}(t) \cdot R$$

Thus we can write:

$$E = \frac{1}{2} m \dot{a}^2 R^2 - \frac{GMm}{aR}$$

Now we can study the 3 possible different universes in function of E. Let’s start with “**E = 0**”. We have:

$$\frac{1}{2} m \dot{a}^2 R^2 = \frac{GMm}{aR}$$

We can cancel “m” in both sides and rearranging we get:

$$\dot{a}^2 R^2 = \frac{2GM}{aR}$$

Notice that like before we can try to get an expression of volume in the denominator so that it doesn’t depend on the location any more. Dividing by “R^2” and “a^2” in both sides we get:

$$\frac{\dot{a}^2}{a^2} = \frac{2GM}{a^3 R^3}$$

As before, the volume of the sphere will be "$\frac{4}{3}\pi a^3 R^3$", so multiplying both the numerator and the denominator by "$\frac{4}{3}\pi$" we get:

$$\frac{\dot{a}^2}{a^2} = \frac{\frac{8}{3}\pi G M}{\frac{4}{3}\pi a^3 R^3} = \frac{\frac{8}{3}\pi G M}{V}$$

As previously we can replace the mass divided by the volume by density:

$$\frac{\dot{a}^2}{a^2} = \frac{8}{3}\pi G\rho$$

Expressing the density in terms of a we get:

$$\frac{\dot{a}^2}{a^2} = \frac{8}{3}\pi G\frac{\nu}{a^3}$$

Now we have a first order differential equation that we can solve. Notice that as before we set the constants to 1, so we get:

$$\frac{\dot{a}^2}{a^2} = \frac{1}{a^3}$$

Rearranging:

$$\dot{a}^2 = \frac{1}{a}$$

Now taking the square root:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{1}{\sqrt{a}}$$

Multiplying both sides by "$\sqrt{a}$":

$$\sqrt{a}\frac{\mathrm{d}a}{\mathrm{d}t} = 1$$

Now we can integrate both sides by “dt”, so we get:

$$\int \sqrt{a}\ \frac{\mathrm{d}a}{\mathrm{d}t}\ dt = \int 1\ dt$$

Canceling the “dt”:

$$\int \sqrt{a}\ da = \int 1\ dt$$

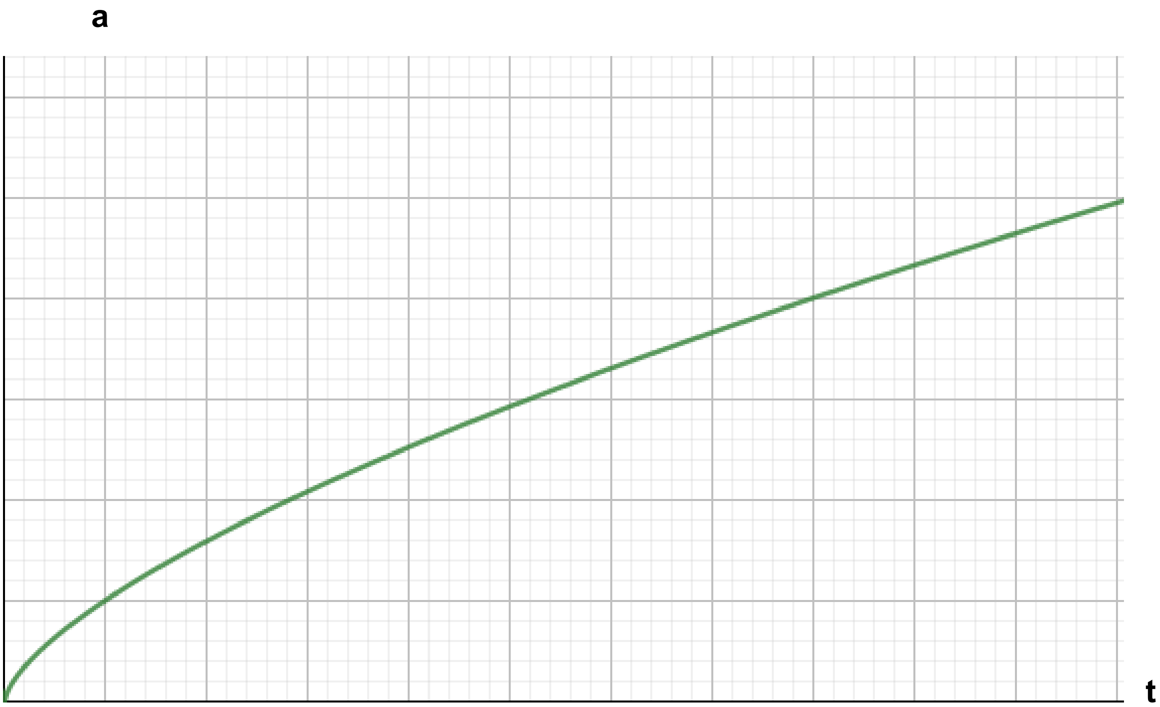
Now we can integrate in both sides to get:

$$\frac{2a^{3/2}}{3} = t$$

Rearranging, ignoring the constant of integration, and setting the other constants equal to 1 for simplicity we get:

$$a = t^{\frac{2}{3}} \left(\frac{3}{2}\right)^{\frac{2}{3}} \quad a = t^{\frac{2}{3}}$$

Now we can graph it:



The x axis is the time and the y axis is the value of a.

Let's analyze the result we get. We have set the total energy equal to "0". As we see the slope of a, namely the time derivative of a or the velocity of expansion, slows down as "t" increases, that's due to gravity. As we see in the infinite the slope will be equal to "0" so we would have a frozen universe. We can prove that by taking the limit of the time derivative of a to infinity:

$$\lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}t} t^{\frac{2}{3}} = \lim_{t\to\infty} \frac{2}{3\sqrt[3]{t}} = 0$$

We see that we just get a universe that will expand forever at a slower rate each time until it becomes a static universe in the infinity. That's the analogous of a body going to the escape velocity, it will go to a slower velocity each time until it has "0" velocity with respect to the body but it won't fall back.

3.2.4. Different universes in function of energy.

Now let's look at the case of a universe with **positive energy**, as we will see the equations for both universes with positive and negative energies will be similar.
Let's go back to the initial kinetic and potential energies, but instead of setting their energy equal to "0" we will set the total energy to be positive. Getting back the equation:

$$E = \frac{1}{2}m\dot{a}^2R^2 - \frac{GMm}{aR}$$

Now note that "E" is a constant. For simplicity we will again set constants to be equal to "1". Dividing by the galaxy's mass in both sides we get:

$$\dot{a}^2R^2 - \frac{2GM}{aR} = \frac{2E}{m}$$

Now we will divide both sides by "a^2R^2":

$$\frac{\dot{a}^2}{a^2} = \frac{2GM}{a^3R^3} + \frac{2E}{ma^2R^2}$$

Now note that "E", "m" and "R" are constants, we can substitute them by some parameter "k":

$$\frac{\dot{a}^2}{a^2} = \frac{2GM}{a^3R^3} + \frac{k}{a^2}$$

Where "k" equals:

$$k = \frac{2E}{mR^2}$$

Notice that neither the term in the left side nor the first term in the right side depend on “R”, since we can substitute “R” as before in the first term in the right side by the volume and obtain the density which doesn’t depend on “R”:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho + \frac{k}{a^2}$$

Since “k” must be independent of “R” we can say that “E” is proportional to “R^2”. In fact in a homogeneous universe the total energy must be constant for a given galaxy and directly proportional to the square of the distance “R” (the number of grids between two points). To see why we can just go back to the equation of the total energy of the universe and check how the total energy changes in function of “R” in a given time. Since in a given time a and “$\dot{a}$” will be constant we can set them to be equal to some constant, so we can write:

$$E = \mu R^2 - \frac{\lambda M}{R}$$

Where both “μ” and “λ” are constants. Now notice that we had an expression for the mass in terms of “R”, substituting we get:

$$E = \mu R^2 - \frac{\lambda \nu R^3}{R}$$

Since “ν” is another constant we can set it equal to “1”. Now notice that the “R” in the denominator will cancel with one of the “R” in the numerator, so we will get:

$$E = \mu R^2 - \lambda R^2$$

Rearranging it we get:

$$E = R^2(\mu - \lambda) = c \cdot R^2$$

Since “$\mu - \lambda$” was just another constant we have set it to be equal to “c”. Keep in mind that “c” is just some arbitrary constant and it has nothing to do with the speed of light.

We can also see that “k” won’t be time dependent, since the total energy “E” will be conserved and “R” by definition doesn’t depend on time.

So we now have the complete Friedmann equation:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho + \frac{k}{a^2}$$

Or substituting “ρ”:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\nu}{3a^3} + \frac{k}{a^2}$$

Notice that for “a” small value of “a” in the early universe the first term will be dominant since “a^3” will be smaller than “a^2” for “a < 1”, and a is in the denominator. When the universe expands and a increases the dominant value will be “a^2”.

Note that the only thing that can change the sign of “k” is “E” since “m” is a mass and “R” is just the value of the distance in grids and will always be positive, so “k” will always be positive for a positive value of “E” and negative for a negative value.

If we derived the Friedmann equation from general relativity we would derive a similar equation:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

Where “c” stands for the speed of light. Remember that in the last equation we derived “k” was some constant that depended on energy, but didn’t have any further physical meaning. In fact notice that if we set “k” equal to:

$$k = -\frac{2E}{mc^2}$$

We would get the same equation. But from Newtonian physics “k” is just some constant, the fact that “c” appears in the equation is just the consequence of some arrangements we’ve made.

But in general relativity the constant “k” has a physical value, it stands for the curvature of the universe. Here we find one of the limitations of newtonian mechanics, we have been able to get some kind of “k” constant dependent on the total energy of the universe, but it’s just a constant we have invented, that’s because Newtonian mechanics don’t have such a concept as the curvature of space, it’s when we derive it from Einstein’s field equation that “k” has a true meaning. I’ll explain later what curvature could our universe have, and how we can know it, but for now let’s just analyze how curvature can affect the expansion of the universe. If we have a positive “k”, namely a positive curvature, we would have a negative energy and consequently the expansion would reverse and the universe would collapse. If there was no curvature in the universe, that is to say “k = 0”, the second term in the right side of the equation would disappear and we would just get back the first equation we derived. Finally if “k” is positive we would have a universe that would expand forever, in this case note that it would start as the case where “k = 0” but at some point it would end in a straight line that would keep growing forever. Let’s derive the equation for both cases to have the value of “a” in function of time. We know that at first the predominant term will be the first one so we will have a growth give by
“$a = t^{\frac{2}{3}}$”, at some point the second term will be the dominant one. For simplicity we will just study the second term of the equation given that we know how the first term will evolve, we could try to solve the whole equation, but it would have a lot of terms and we wouldn’t really get anything we won’t get by just solving the second term of the equation. Setting c equal to 1, when the a factor is large enough we will have:

$$\frac{\dot{a}^2}{a^2} = k$$

Canceling a in both sides of the equation we get:

$$\dot{a}^2 = k$$

Now we can take the square root in both sides:

$$\frac{\mathrm{d}a}{\mathrm{d}x} = \sqrt{k}$$

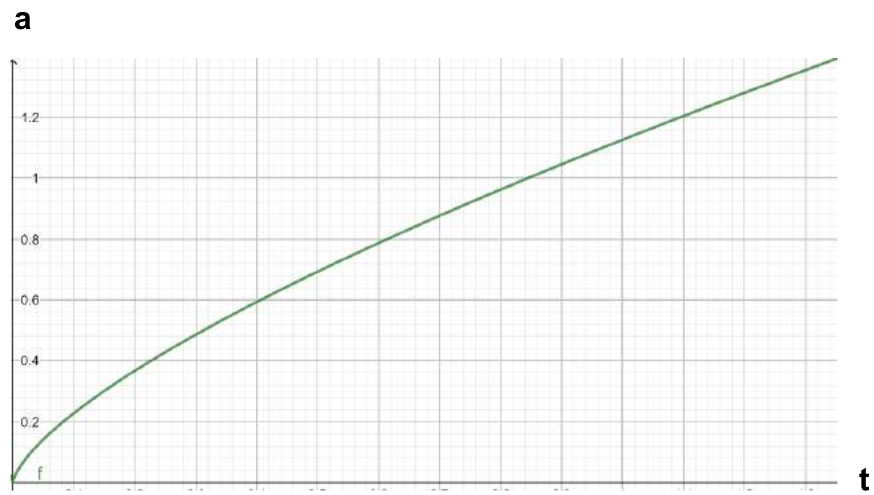
Finally we can rearrange it and integrate in both sides:

$$\int da = \int \sqrt{k}\ dt$$

Note that the square root of a constant is another constant, so we can substitute "$\sqrt{k}$" for "k", we get:

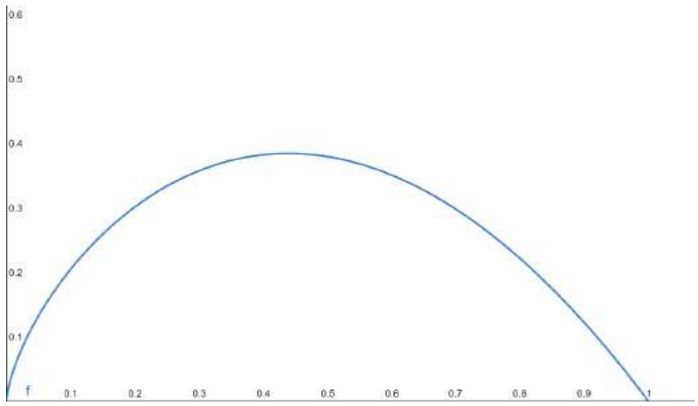
$$a = kt$$

So we know a universe with positive energy, as we have seen, is equal to saying a universe with negative curvature, must start like a universe with 0 energy, until at some point the second term becomes dominant and ends up in a straight line. It might seem similar to the case of the universe with 0 energy, but in this case the universe will keep growing forever in a straight line with constant slope. Note that this would be the analog case of a rocket going faster than the escape velocity, at some point gravity would be so low that it would just keep going at a constant velocity.



Graph showing how a universe with positive energy would evolve, the x axis is the time and the y axis is the value of a

From this analogy we can guess how the negative energy case will evolve, it will be analogous to the case of a thrown apple that falls back.



Universe with positive curvature/negative energy that will colapse

Now we might ask ourselves how's our universe? As we've seen, whether our universe will keep expanding or will collapse depends on its curvature. We will discuss later how we can know its curvature, but for now we will analyze something different.

By now we have seen examples of a matter dominated universe, however relativity teaches us that there's some kind of equivalence between mass and energy. That's how we introduce the radiation dominated universe, when we derive the Einstein

Field equations for the universe we will see how a radiation dominated universe would behave.
As we have said, there might probably be 3 curvatures for our universe, and as we have seen the curvature affects the expansion. The first universe we have seen is the “K = 0” curvature one, this case would be analogous to the “0” energy universe. A “0” curvature would mean that our universe is flat in 3 dimensions, in this case if we had a triangle the sum of its angles would be equal to 180 degrees, that’s in fact how it looks in small scales, we can imagine it as a giant cube with grids. In this case the universe could be infinite since it could extend forever, although some scientists have also hypothesized a non infinite flat universe without boundaries, we can visualize it as the boundaries of a game such as PacMan, when you get to a boundary you would appear on the other side, you could imagine some kind of torus, no matter where you go, along any axis you would go back to the same point. One could say that a torus isn’t flat, and in the analogy of a doughnut that would be true, however in a 4D torus it’s possible to have a flat 3D space.

But we still could have two other possibilities. Both cases are difficult if not possible to visualize in three dimensions that’s why we will use analogies in two dimensions. The first one would be a universe with positive curvature, we might imagine it as a three dimensional sphere, where we would live in its surface as two dimensional beings, in this case the angles of a triangle would sum more than 180 degrees and we would be able to get back to the same point with just turning 90 degrees to one side 3 times. In this case since the geometry of the universe would be affected, let’s imagine ourselves, as flat squares over a pole, as things get away from us we would see them bigger, until they would reach the antipode where they would look huge. So if we count the number of galaxies we see we would notice that the further away we go, the less galaxies we would see until in the antipode we would just have the chance to see a single one that would look huge. This universe would always be finite without boundaries.

Finally there’s the negative curvature universe called hyperbolic, this one is probably the most difficult to visualize. We must go back to the bidimensional world. Think about Pringles snacks, or maybe a horse saddle. Here the angles of a triangle would sum less than 180 degrees, and we would need to turn 90 degrees to one side 5 times to get back to the same position. As before the geometry would be affected too, the further an object is the smaller we would see it and unlike before there wouldn't be such a thing as an antipode. Again we could tell whether our universe is hyperbolic or not by counting galaxies, as we looked further we would see more galaxies. As in the flat universe, an hyperbolic universe could be infinite or as before it could be made of some kind of 4 dimensional torus, in such a case when you arrive at a boundary you would appear on the other side.

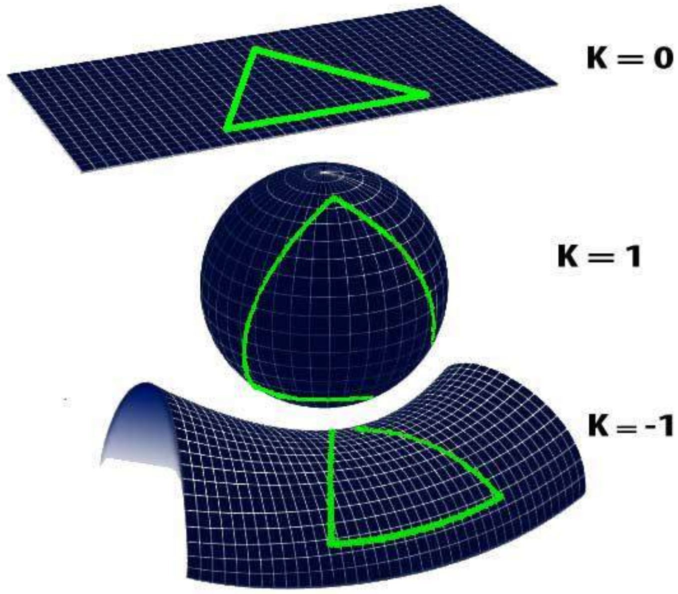
As we will see, in general relativity matter and energy (as we will see they are equivalent) tell the space how to curve, in fact another way to know whether we live in a flat, or another universe is by calculating its density. Let's go back to the "0" energy universe:

$$\frac{\dot{a}^2}{a^2} = \frac{8}{3}\pi G\rho$$

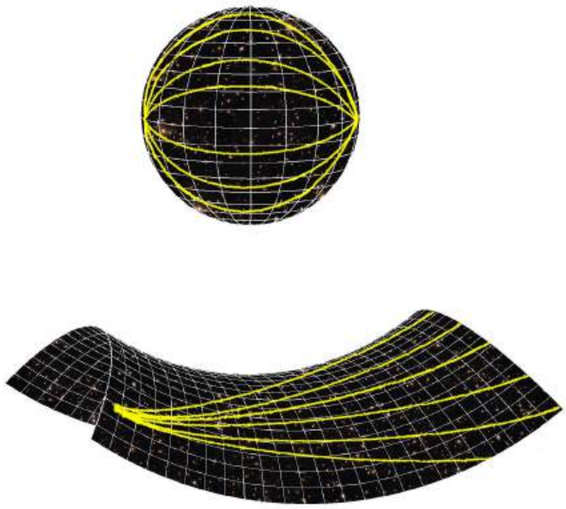
Remember that "$\frac{\dot{a}}{a}$" is indeed the Hubble constant, and in the right side we have the mass density, so rearranging it we can get an expression for the density in a flat universe:

$$\rho = \frac{3H^2(t)}{8\pi G}$$

Notice that for a positively curved universe, the second term on the right side will turn positive on the other side, so the total density will be higher, on the other hand for a negatively curved universe the second term will be negative on the other side, so the total density will be lower. So by calculating Hubble's constant we can determine using the last equation what's called the critical density, which it's an expression that tells us what density the universe needs to be flat, if the total density is lower it will be hyperbolically curved, if it's higher it will be positively curved.



Universes with positive, 0 and negative curvature represented in two dimensions



As we see in a spherical universe when we look far enough we end up seeing the same point at every direction, while in a negatively curved universe we can see a lot of objects by just looking at one direction.

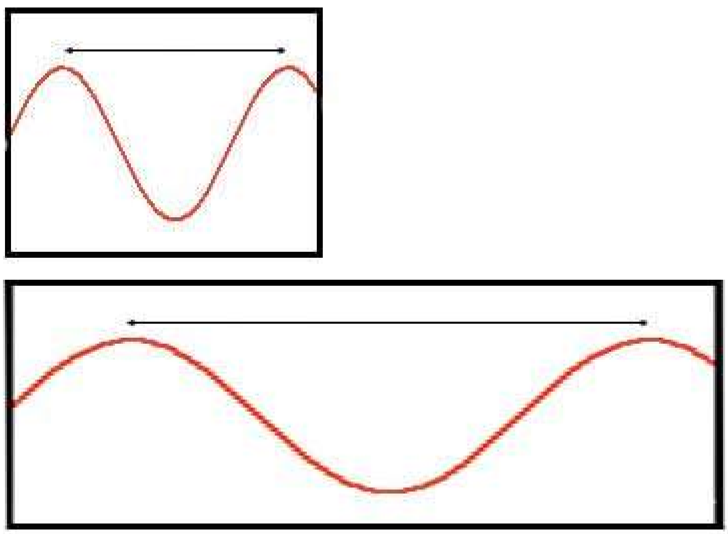
3.2.5. Radiation dominated universe

Until now we have studied the cases for a matter dominated universe. In special and general relativity we learn that mass is equivalent to energy, so from general relativity we would get a Friedmann equation with energy density instead of mass. That would allow us to study other universes which are filled with energy such as the radiation dominated universe, where photons would dominate the energy of the universe, in fact we believe that was our early universe.

In this case the energy density would behave differently, that is because the energy of a photon depends on its wavelength:

$$E = \frac{hc}{\lambda}$$

Where h is Plank's constant, c is the speed of light and λ is its wavelength. Note that in an expanding universe the wavelength increases since the "grids" are expanding. We can imagine it as if the light wave was attached to the walls of a box, like the string of a guitar. When the box expands, the wavelength increases.



The wavelength increases because of the expansion of the universe

So we find that as the universe expands, the energy of the photon decreases:

$$E \propto \frac{1}{a}$$

That "loss" of energy causes the wavelengths to be redshifted, that's the cosmological redshift we will calculate. With that, the Friedmann equations will now depend on a higher order of a, since the density will now be proportional to $\frac{1}{a^4}$, which would have made our early universe expand faster.

3.3. The theories of relativity

Now we have a good understanding of the universe using Newtonian cosmology, but we can still go further applying the general theory of relativity, but for that we will first need to do a brief introduction on both special and general relativity, they are a really complicated topic, especially general relativity. I don't expect the reader to understand everything deeply but to have a basic understanding on the topics of special and general relativity with the use of maths, so the meaning of the concepts of the most important equations can be understood. The next pages will be based on Leonard Susskind's lectures on both special and general relativity, as well as on the information provided by "L'astronomica de Sabadell". For general relativity I will also use the information provided by the books "*Gravity - An Introduction to Einstein's General Relativity - J. Hartle (Pearson, 2003)*" and "*Carroll, S. M. (2019). Spacetime and geometry: An introduction to general relativity. Cambridge University Press*".

3.3.1. A bit of history

At the end of the XIXth century physicists though they could describe almost everything in the universe, they thought it was just a matter of time to be able to describe any phenomena in the universe with our equations, with a powerful enough computer we should be know the trajectories of every molecule and predict anything in the deterministic universe, but there were some experiments and phenomena that couldn't be explained, which were seen as small clouds in the horizon that would soon be explained and disappear. In fact there's an anecdote that tells us that one of Planck's teachers told him to choose another branch of science since there wasn't much else to discover in physics. Fortunately Planck ignored him and became the father of quantum mechanics. It turned out that these small clouds in the horizon became the basis of two totally new and revolutionary branches of physics called Relativity and Quantum Mechanics. Without them, technologies such as the GPS, nuclear bombs, transistors and consequently microprocessors wouldn't exist. If you are now able to search anything on your phone or to contact anybody in the world almost instantaneously is thanks to these clouds that looked tiny but were huge.

For the purposes of this project we will just talk about the cloud that lead to the discovery of the general theory of relativity, as we will see we will have to leave our common sense and start accepting that most things are not how they appear to look like, we will travel to a world were time and space are the same thing, where people can time travel (to the future), where one twin ages faster than the other.

In the last years of the XIXth century Maxwell unified the theories of magnetism and electricity with just 4 equations. They set the end to the development of what we now call Classical Mechanics and they appeared to describe electricity perfectly, they just seemed to have a problem, they required the light to travel at the same speed for any frame of reference and this didn't seem to be possible according to Newtonian physics. But a law of physics must be the same in every inertial frame, that means that in a frame with constant velocity where no forces are acting over it the laws of physics should be the same, that is to say that if the theory of electromagnetism says that light should always move at the same speed for any inertial frame, it should move in the same way. Let's analyse it from a Newtonian point of view.

3.3.2. The Newtonian coordinate transformations

Suppose we have two inertial frames, one of which is at rest with respect to the other. In fact, notice that speed is relative, since there are no forces acting on a moving frame of reference (if it's moving at a constant speed), the person who's moving can say he's at rest and it's the other frame of reference who's moving. In the end how can you tell whether you are moving or not without comparing your position to something? The answer is that there's no way, to know it where the idea of classical relativity comes from. We will call the "moving" reference frame "X' " and the static reference frame "X", so "X' " is moving relative to "X" at a velocity "v". For simplicity we will say that "X' " moves along the x axis. Suppose we are in the "static" reference frame and we see an object, we could describe its position by a distance " Δx". So that if an object is 2 meters away from us along the x axis, we would describe its position as "x = 2". Now think about the person in the moving frame of reference, he can say he's static since as we have said there are no forces acting over him, but he sees the object is moving relative to him, so in his reference frame the object won't be described by "x = 2", furthermore its position will depend on time. If we are in the "static" reference frame, how can we describe the position of the object in the coordinates of the moving person? That means, we see the object from our static reference frame, but we want to know how the person in the moving reference frame will see it. We need some kind of coordinate change, so that by observing a position in my frame I can translate it in terms of the moving frame.

Let's first describe the motion of an object in the moving reference frame from our frame, we can describe it as:

$$X = vt$$

Or subtracting "vt" on both sides:

$$X - vt = 0$$

But the person who's in the moving frame will always say she is in the center of coordinates and since no forces are acting on him he can say we are the ones who are moving, so he will describe his position as:

$$X' = 0$$

Now we can represent the coordinates of the moving reference frame from our coordinates:

$$X' = X - vt$$

Or:

$$X = X' + vt$$

Now we can change the reference. The person who's moving can say he is standing still and it's the object and us who are moving but on the other side. So he will write:

$$X' = -vt'$$

Or:

$$X' + vt' = 0$$

Where t' is the time for the moving frame of reference. As done, since we are always in the center we will describe our position as:

$$X = 0$$

So he would represent our coordinates from his coordinates:

$$X = X' + vt'$$

Now Newton would have supposed that the time is absolute for all observers, so the time for an observer would be the same for the other one, because that's what we see in the real world. Following this principle, now the expressions of the moving frame and our expression to change of coordinates would coincide, so according to Newton we would have the next coordinate transformations:

$$t' = t$$
$$X = X' + vt$$

Let's suppose we, in the "static" frame of reference, see a light ray, its position will be described by "$X = ct$", where c stands for the speed of light, substituting in the last equation:

$$ct = X' + vt$$

Now let's see what the moving reference frame would see. We just have to subtract "vt" in both sides:

$$ct - vt = X'$$

Rearranging:

$$(c - v)t = X'$$

There's clearly something wrong here, the observer in the moving frame of reference would see that light is moving slower than "c". We could now change the direction so that we can describe its movement by "x = -ct". It's easy to see that now the other frame of reference would describe the movement of the light ray as:

$$-(c + v)t = X'$$

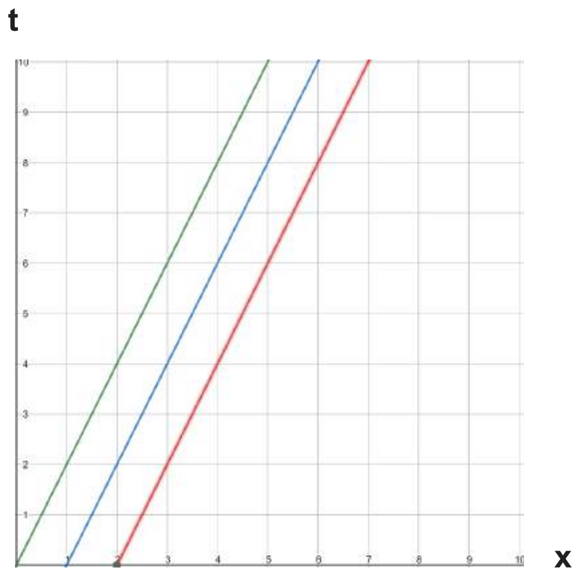
So he would see it moving faster than light. But no forces are acting on it, thus the laws of physics must be the same for every reference frame, in fact we have already seen that the movement is relative, it doesn't make any sense to say someone is moving if you are not comparing it to something else. So there were two options, either Newton (the father of modern physics) was wrong, or Maxwell's laws of electromagnetism were incorrect. Physicists tried to explain it without any result, many physicists hypothesised that light moved through some kind of fluid called ether and that when we moved we would see light moving slower, hence hypothesising a privileged reference frame where the ether was at rest. There was just a problem, the earth was moving relative to the sun, so we should be able to measure how the light moves slower because of the earth movement relative to the hypothesised ether. The theory of the ether sounded good, there was just one thing left to do, to prove that because of the movement of the earth we should measure light moving slower than c (the value for the velocity of light given by Maxwell's laws). A lot of experiments were done, such as the Michelson-Morley experiment, every time improving their precision, but all of them measured the same velocity c for light. Something seemed wrong and nobody was able to solve the problem, the theory of the ether seemed a good way to solve it and was elegant, in fact a physicist called Lorentz even derived the true equations/coordinate transforms, supposing that when moving, the ether would cause some kind of pressure, but he just saw them as some kind of approximation.

3.3.3. Space - time diagrams

Finally a young physicist called Einstein would figure out the problem with his PhD thesis called "*On the electrodynamics of moving bodies*". He found a way to make the speed of light an absolute value without needing the existence of an ether or a preferred reference frame, he found a way to make the speed of light be the same for every reference frame, there was just one problem, he was proving Newtonian physics were wrong. At first he was ignored, but with time people started paying more attention to his idea, and realized he was indeed right. That's how special relativity was born. Let's see what Einstein did discover.

Let's think about two frames of reference, one of them is moving at a high speed relative to the other. For simplicity from now we'll describe velocities in terms of dimensionless fractions of the speed of light "$v = \frac{V}{c}$", such that "c = 1", we will also define distance in light seconds (the distance a light ray travels in one second) . We will have to take it into account later to transform our equations to have consistent units. We are again in the "static" frame of reference. In the other frame of reference there's a train moving at a constant speed. There are 3 passengers in the train, Nico, Sofia and Anna, each of them is in a different rail car and each rail car is separated by a unit of distance.

We will draw a graph of their position in function of time. They are common in special relativity, they are called space time diagrams. Let's suppose the moving frame of reference and our frame of reference start at the same point and we synchronise our clocks, such that at the beginning we have "$X = 0$", "$X' = 0$", "$t = 0$" and "$t' = 0$". From our frame we will draw the following diagram:



Space time diagram, the horizontal axis represents the position and the vertical axis represents time

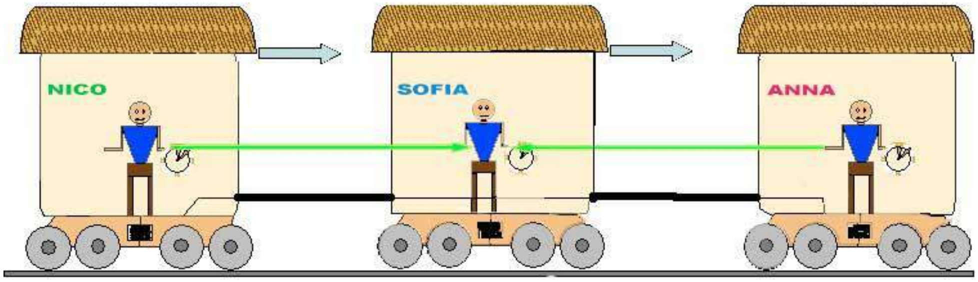
Suppose we see a light ray in our frame of reference it will move according to the equation:

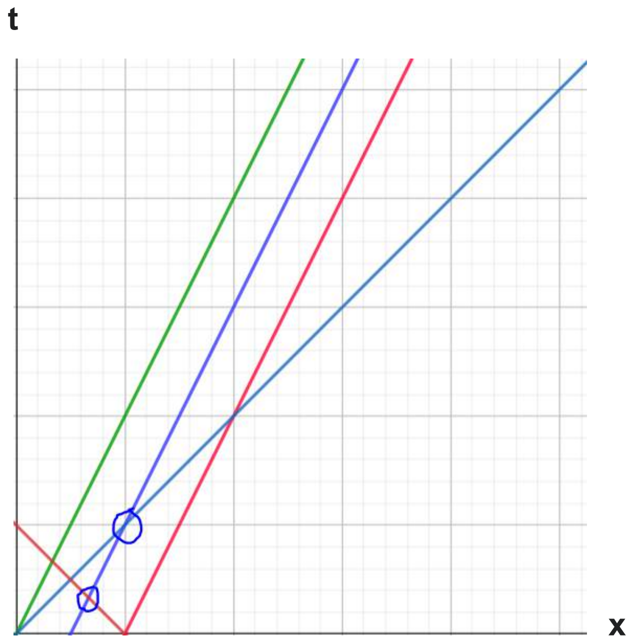
$$x = ct$$

Since we have defined our units so that the speed of light is equal to 1, we can describe it with:

$$x = t$$

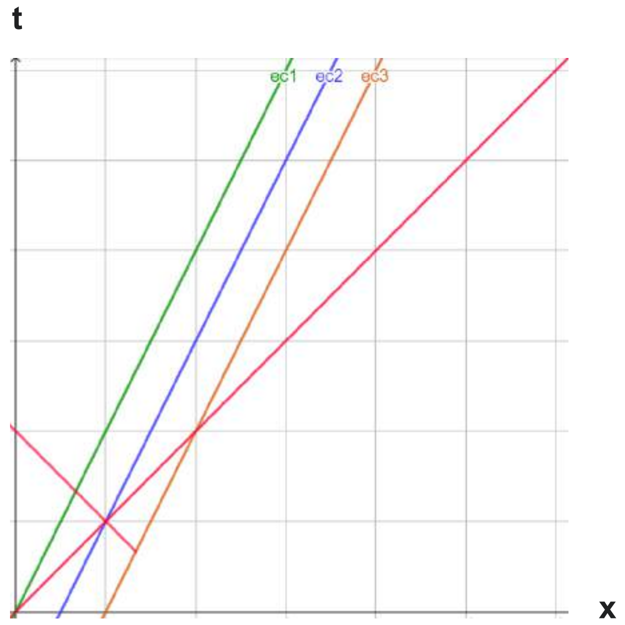
That tells us that in the units we have chosen, light will always move at an angle of 45 degrees. And since every reference frame sees light going at the same speed, they will also see it moving at an angle of 45 degrees. Now let's suppose that Nico and Ana send a light ray to Sofia at the same time, since they are in a reference frame and both are at the same distance from her, Sofia will see that both light rays arrive at the same time, but that won't be true in our reference frame. Let's take a look. As we said before, in our units, light will always move at an angle of 45 degrees, let's graph the two light rays, each going to the opposite direction, from our reference frame:





Space time-diagram showing that while in one reference frame two events are simultaneous, they may not be simultaneous in another reference frame.

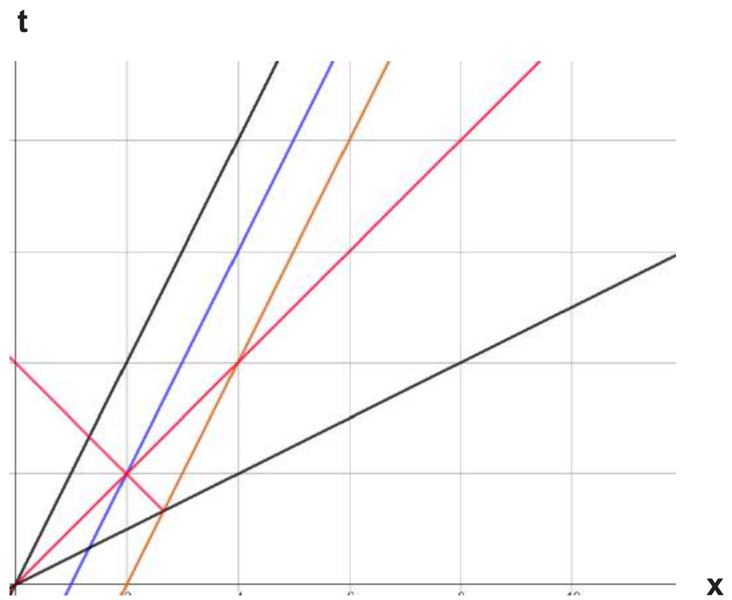
We see that two events that were simultaneous in one frame aren't simultaneous in our reference frame, in fact in our reference frame to make the light rays arrive at the same time we would need that Ana sent her light ray later, so that we get the following graph:



Now Anna sends the light ray later so that it arrives at the same time as Nico's light ray

In order to see both rays arriving at the same time in our reference frame Ana would have to send her light ray later, but in the moving reference frame both will be sent and will arrive at the same time, that's how we can conclude that the time and position axis must be different in the other reference frame, let's see how.

First of all remember that the people in the train are in a reference frame, that means that they can say they are not moving since there are no forces acting on it, so while in our reference frame we see they are moving from the origin in their reference frame they will say they are always at the origin, so their vertical axis must be the line with the one which represents the position of the train, supposing that they chose their origin to be at the back of the train so that it coincides with the "static" frame. In our case it will coincide with the line describing Nico's position in function of time. So we see that the vertical axis, the line where all points at the origin of coordinates are located, is shifted. What about the horizontal one? We know that events that occur at "t = 0" are always located in the horizontal axis, so two simultaneous events must be located either in the horizontal axis or in a line parallel to the horizontal axis. Let's think about the light rays, in the static reference frame they were not simultaneous, but in the moving reference frame they must be, so the horizontal axis must go through the two points in the graph where Nico and Anna send their light rays respectively.



Graph representing the coordinate axes of the moving frame from the "static" frame

We have graphed the coordinate axis of the moving frame from our coordinate axis, this coordinate axis seems to be valid, notice that in both coordinate frames light will move at the same velocity since it will travel the same distance in the same amount of time given that in both cases it describes a bisector. Now from the functions of position of the travelers in the train we could derive an expression for how time and position change in the moving frame of reference, but we won't do it here, instead we will follow the general derivation for any moving bodies Einstein did in his paper.

3.3.4. Derivation of Lorentz

So far, we have seen that the newtonian changes of coordinates aren't valid when we say that the speed of light must be the same for every observer, now we have seen that when we describe a moving frame of reference it's coordinate axes rotate in opposite directions, such that their angle is smaller as seen from the "static" frame of reference. It's easy to see that the previous transformations are no longer valid because they supposed that time was something absolute, but if the axes describing position and time move, we can conclude that time must be different for two different reference frames. We also know that the change is only noticeable if the observers are moving fast enough, since it's something we don't see in our lives with slow moving reference frames. So there must be some kind of function that changes the coordinate axes in function of the relative velocity, and that when we move in slow speeds it becomes negligible. Finally, the galilean coordinate transformations are invalid, but they get a thing right, that is that "$X' = 0$" whenever "$X = vt$", so when describing the position of an object at the origin the change of coordinates will work. So now we can write our coordinate transformations such that it retains that property:

$$X' = (X - vt)f(v)$$

And for time, since when "$t' = 0$" whenever "$t = vX$", following a similar reasoning as before, we can invert the roles of "x" and "t" and write:

$$t' = (t - vX)g(v)$$

This doesn't seem to be consistent, but as we will show later, if you take into account the units we are using, it is. So we see that both equations are similar, they have some kind of symmetry at the origin of the moving frame:
"$X' = 0$" whenever "$X = vt$" and "$t' = 0$" whenever "$t = vX$", because of this symmetry they tell us that the "t' "axis is just a reflection of the "x' " axis about "x = t", that is indeed what we have seen in our space time diagrams.

Now notice that whether the velocity is positive or negative, that is to say whether we are moving in one direction or another the effects on "x" and t will be the same, in physics nothing requires movement to one direction to be represented as positive, so the functions in our coordinate transformations must not depend on whether the velocities are positive or negative, so we can write them as functions of the square of the velocity, since a negative number square returns the same as the opposite number squared. So we now write our transformation equations as:

$$X' = (x - vt)f(v^2)$$
$$t' = (t - vx)g(v^2).$$

Now let's think about a light ray, as we have said light must travel at the same speed for every observers, so if in our units the speed of light "c" is equal to 1 in one frame of reference, it must be equal to 1 in the other frame of reference. So if we can describe the movement of a light ray in one frame of reference as:

$$X = t$$

We must be able to describe the position of the light ray in the other frame as:

$$X' = t'$$

So if we use the transformation equations for a light ray, setting "$X = t$" and requiring that "$X' = t'$ " we end up getting that:

$$f(v^2) = g(v^2)$$

So the fact that the speed of light must be the same for every observer leads to the requirement that $f(v^2)$ and $g(v^2)$ are the same function, so we write:

$$X' = (X - vt)f(v^2)$$
$$t' = (t - vX)f(v^2)$$

From the galilean invariance, we can say that every inertial moving frame is "static" and it's everything that moves, that means that if we are in an inertial frame we can't tell whether we are moving towards something at a constant speed or it's the other object who is moving towards us at the same speed and inverted direction. So the function that relates the two frames of reference has to be symmetrical, the only difference is that the velocity is the opposite, so we can write the inverse transformations as:

$$X = (X' + vt)f(v^2)$$
$$t = (t' + vX)f(v^2)$$

Now we have two sets of equations we have derived following Einstein's principles, we can check if they are compatible by plugging one of the equations into the other, that means taking the last equations and plugging in the equations for "X' " and "t' ". Then we will require to get back "$X' = X$" and "$t = t$", from there we will be able to cancel the "X" and find a valid expression for "$f(v^2)$". Starting with the first equation:

$$X = ((X - vt)f(v^2) + v(t - vX)f(v^2))f(v^2)$$

For simplicity we will substitute "$f(v^2)$" by "f":

$$X = (X - vt)f^2 + v(t - vX)f^2$$
$$X = Xf^2 - vtf^2 + vtf^2 - v^2Xf^2$$
$$X = Xf^2 - Xv^2f^2$$
$$X = Xf^2(1 - v^2)$$

Now we can cancel "X" on both sides and solve for "f":

$$f(v^2) = \frac{1}{\sqrt{1-v^2}}$$

So now we have an expression for the function that satisfies all the conditions we had set. Let's substitute "$f(v^2)$" in our equations:

$$X' = \frac{X-vt}{\sqrt{1-v^2}}$$
$$t' = \frac{t-vX}{\sqrt{1-v^2}}$$

Now, note that we have been using relativistic units (they are really called Planck units). To make the transformations have consistent units. In the first case note that in the numerator we just have units of position, so we don't need to do anything to restore the international system units, however in the denominator we have meters per second, keeping in mind that in our definition v was dimensionless since it was defined as "$\frac{v}{c}$", we see that restoring the "c" in the denominator we have an expression with consistent units:

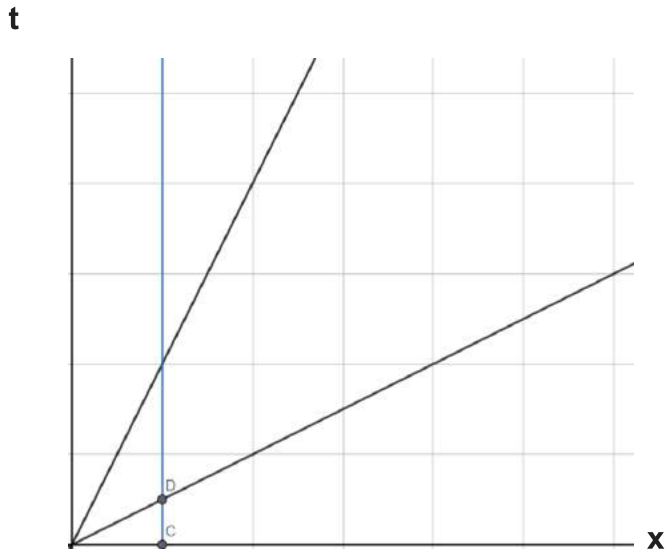
$$X' = \frac{X - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

In the expression for time in the numerator we have vX which is meters squared over seconds, we can make it have time units by dividing by "c". In the denominator we can do the same thing as before, we get the following expression:

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}}$$

These are the Lorentz Transformations, they describe how the coordinate axis changes compared to another reference frame, let's see how they work.

Let's imagine I'm in a reference frame and you are moving relative to me, so I see you are moving at a constant speed. I am holding a stick that measures 1 meter, but you are not so sure about it, at least you shouldn't be. Let's draw a space time diagram:



Spacetime diagram, the blue line shows where x = 1

As we see at “t = 0” I will see that the stick starts at the origin and ends at point “C”, that is one meter away from the origin, but if you measure the position of the two ends of the stick at the same time, you will see that the sticks ends at point “D”, since it’s where your time axis equals “0”, but notice that the length won’t be the same. Let’s see how in our equations. In relativistic units we have:

$$X' = \frac{X - vt}{\sqrt{1-v^2}}$$

And we know that point “D” is in the interjection between “x = 1” and “$t' = 0$”, and from our transformations we know that “$t' = 0$” whenever “$t = vx$”, so we can now substitute to get:

$$X' = \frac{1-v^2}{\sqrt{1-v^2}}$$

Or:

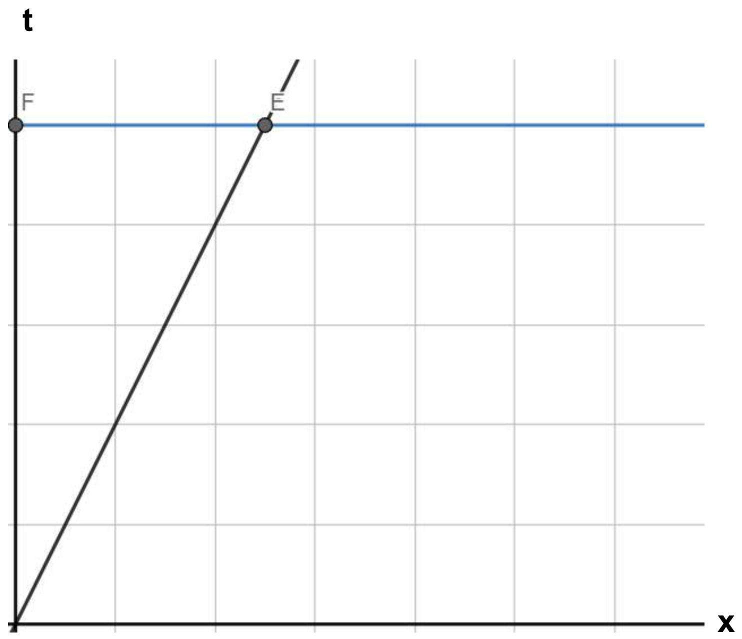
$$X' = \sqrt{1 - v^2}.$$

That’s it! We now have an expression for the length of the stick for the “moving” frame of reference supposing the stick measures 1 in the “rest” frame. That’s called length contraction. Let’s substitute v by a considerable fraction of the speed of light, such as “0.5 c”:

$$X' = \sqrt{1 - 0.5^2} \approx 0.87.$$

The stick has shortened! The closer the speed relative to the “static” frame is to light, the shorter the stick will appear to be. Now imagine that you hold the stick, you say you are not moving since there are no forces acting on you and it’s me who is moving relative to you, following the same reasoning as before, since I’m moving relative to you I will also see that your stick is shortening! So the same equation would be valid to calculate the length of the stick from my reference frame. So both observers see that the stick of the other one is shortening.

And what about time? Let’s think about clocks, you are moving relative to me wearing a clock at uniform velocity, suppose that I have another clock synchronized. When your clock reads 1, what will be the time in my frame? Let’s draw again a spacetime diagram:



Spacetime diagram showing the t’ and t axis

The points placed on the horizontal line are what I would call synchronous. Your clock is moving along the t’ axis which is represented by “$X' = 0$” and we know that you are measuring "$t' = 1$", to figure out what time I will see in my clock when your clock measures "$t' = 1$" we just have to use the Lorentz transformation equations:

$$t = \frac{t'+vX'}{\sqrt{1-v^2}}$$

Substituting "$t' = 1$" and "$X' = 0$", we get:

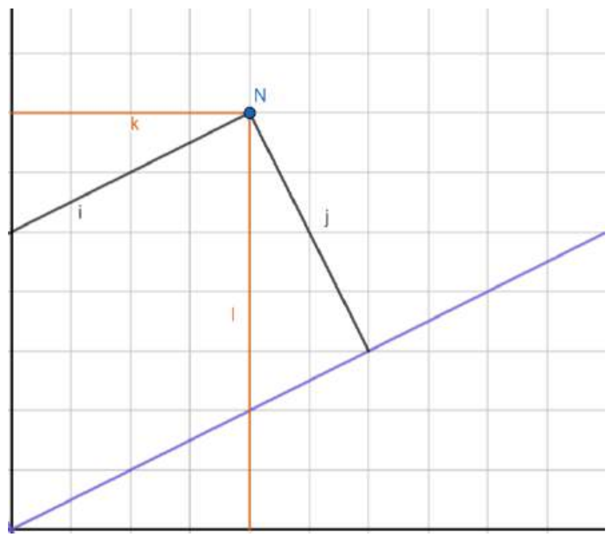
$$t = \frac{1}{\sqrt{1-v^2}}$$

Since at high velocities the denominator will be less than one we see that my time interval will be bigger than the time interval measured by you in your clock. In other words, as viewed from my reference frame your clock will be running slower by a factor of "$\sqrt{1 - v^2}$", but as before, you could do the exact same reasoning and say it's me who is moving relative to you, so you'll see that my clock is slowing down too. One can think about a lot of paradoxes created by it, but they can all be solved by drawing spacetime diagrams, such as the twin paradox, suppose there are two observers in inertial reference frames, and one of them goes to proxima centauri (the closest star) and comes back, for what we have said aren't both observers supposed to see each other slowing down? But if the person going to proxima centauri turns back to the earth to compare their clocks they should agree on who's clock slowed down. The reader can try to draw a space time diagram to solve this paradox.

3.3.5. Proper time and metric

But now there's a problem, the laws of physics must be the same in every place, quantities like energy must be the same for every observer, we need to describe our laws with invariant quantities, that is to say, quantities that don't change when measured from different perspectives, they must be the same for every reference frame. Suppose we want to express a point in space time, is there any way to get an invariant quantity out of it?

Let’s look at a property of euclidean geometry (the geometry taught at school), suppose we have a two-dimensional plane with two sets of cartesian coordinates “x,y” and “x’,y’ ”, and while sharing the same origin the “x’,y’ ” angles are rotated by some angle. Suppose we have a point p, the coordinates “x,y” and “x’,y’ ” will be different, but since both share the same origin they will find that the point is at the same distance, so the quantity “$s = x'^2 + y'^2$”, will be equal to “$s' = x^2 + y^2$”.



Point N is represented by different coordinates but the distance from the origin is always the same, consequently it’s an invariant quantity.

Let’s look at our spacetime diagrams, the coordinate axis for the different reference frames share the same origin, but this time instead of having a rotation, the angle seems to reduce while keeping a symmetry around around the line “$x = ct$”. Can we find some kind of invariant value? We could try to see if the value “$t^2 + x^2$ “ is invariant, we can check it by substituting “X' “ and “t' “ by the transformations in terms of “X” and “t”, but we would get that it isn’t. What if we subtract the squares instead? If our value is invariant it must satisfy:

$$t^2 - x^2 = t'^2 - x'^2$$

Substituting “t' “ and “x'”:

$$t^2 - x^2 = (\frac{t-vx}{\sqrt{1-v^2}})^2 - (\frac{x-vt}{\sqrt{1-v^2}})^2$$

$$t^2 - x^2 = \frac{(t-vx)^2}{1-v^2} - \frac{(x-vt)^2}{1-v^2}$$

$$t^2 - x^2 = \frac{t^2-2vxt+v^2x^2}{1-v^2} - \frac{x^2-2vtx+v^2t^2}{1-v^2}$$

$$t^2 - x^2 = \frac{t^2-2vxt+v^2x^2}{1-v^2} - \frac{x^2-2vtx+v^2t^2}{1-v^2}$$

$$t^2 - x^2 = \frac{t^2+v^2x^2-x^2-v^2t^2-2vxt+2xvt}{1-v^2}$$

The “$2vxt$” terms cancel out, rearranging:

$$t^2 - x^2 = \frac{t^2-v^2t^2+v^2x^2-x^2}{1-v^2}$$

$$t^2 - x^2 = \frac{t^2-v^2t^2}{1-v^2} - \frac{x^2-v^2x^2}{1-v^2}$$

$$t^2 - x^2 = \frac{t^2(1-v^2)}{1-v^2} - \frac{x^2(1-v^2)}{1-v^2}$$

Canceling “$1 - v^2$” we get:

$$t^2 - x^2 = t^2 - x^2.$$

What does that mean? It means that it doesn’t matter at what velocity you move, how the time dilates and the lengths contract, every observer in an inertial reference frame will agree on the value of “$t^2 - x^2$”. Now we have an invariant quantity to make the relativistic laws of physics. Now we can create some kind of metric to represent our points in spacetime as if we were representing points in a two-dimensional plane.

First of all, remember that in an inertial reference frame no forces are acting on it so particles must follow a straight path, that means we can always chose our coordinates such that particles always move along the x axis, but we could add the other spatial coordinates so that we can choose any other coordinates. So we have the following invariant value:

$$\tau = t^2 - x^2 - y^2 \; - z^2$$

That's called a metric with which we can represent the distance in spacetime such that every observer will agree on its value. Why should we care about it? It can tell us a lot about events in spacetime. Suppose that we have an event that occurs at some time "$t \; = \; 1$" and it's at a distance of "$X \; = \; 10$" (we are using relativistic units, where "c=1" and the units of distance are defined as the distance light travels in one second), we can easily calculate the invariant value "τ":

$$\tau = 1^2 - 10^2 \; =- \; 99$$

Note that in this example light doesn't have enough time to travel to the point, so they are spacelike separated, that means that there's no way one can affect the result of the event, no matter at what velocity you go, every reference frame at the same origin will agree on that, and we can know it just by calculating the value of τ.

On the other hand when if we have some event that occurs on "$t \; = \; 10$" and "$x \; = \; 1$" we will see that t will be positive:

$$\tau = 10^2 - 1^2 \; = \; 99$$

That tells us that they are timelike events, it's still possible for a reference frame moving fast enough to get on time before the event occurs, so if we send a light ray from the origin to the event the people in the event won't see the light ray since it will have already passed.

Suppose that in Proxima Centauri an alien party is held, and we as earthlings want to send a light ray to the party such that it arrives when the event starts, because it's when the refrain of a song starts and we are responsible for the special effects. We just need to know at what distance proxima centauri is (it's approximately 4 light years away), and we can calculate at what time our light ray must be send by making "τ be 0". So for an event placed at "τ = 0" we know that a light ray sent from the origin of coordinates will arrive at the time and place of the event.

3.3.6 A bit of general relativity

Now that we have seen how the special theory of relativity changed our way of seeing everything, we are ready to go a step further to study general relativity. In this case the mathematics get too complicated so we will just see some main concepts, for more detail of the theory there will be an annex where the Friedman equations are derived from Einstein Field equations while we will try to explain all the tensors and things that form them.

In the XVIIth century Newton developed his law of universal gravitation, and it seemed to work perfectly for everything. In fact, today most things are calculated using Newton's theory, even when rockets are launched in space we still use his law. However Newton's law seemed to fail in some specific cases, such as Mercury's perihelion (a rotation of mercury's orbit). It was also known from special relativity that mass and energy were equivalent, so that meant that if mass created a gravitational field, energy should behave similarly. Also Newton's laws didn't explain how gravity was transmitted, and from special relativity people knew it couldn't transmit faster than the speed of light.

Einstein thought about the equivalence principle, which states that locally it's impossible to distinguish between a gravitational force or the force caused by some acceleration. He imaginated a rocket accelerating and a light ray passing through its window, he concluded that an observer in the rocket would see it curving, so if locally there was no way to differentiate an acceleration and gravity, a similar phenomenon would occur under a gravitational force, which in fact is the explanation of gravitational lenses.

He also noticed that when we are in free fall due to a gravitational field, there's no way to demonstrate locally whether we are standing still in the space or falling, in fact if we looked at another object under the same gravitational field we would see it falling at a constant velocity, following the laws of special relativity.

Along with the fact that light always moves in the shortest path in space, all this means that when we are in a free fall we are really moving because of inertia and it's the spacetime that curves and makes our path to be the one that falls. That explains that when we are on the surface we feel a force since we tend to follow our inertial movement along space time but there's something preventing us from following it.

That's really hard to understand, but the key point is that General Relativity tells us that gravity can be explained as a curvature of space-time. What causes space-time to curve, well that's the same as asking what causes gravity to occur. The classical answer would have been mass, but in special relativity we realise that mass and energy are equivalent, so it depends on energy, furthermore from this we discover that other factors such as pressure also affect the curvature of space-time (cause a gravitational field).

We explain all this with the einstein field equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

The letters with subindexes represent tensors. Tensors are vectors that are independent from any change of coordinates, they are valid for any observer under any circumstances, and they are the main reason why general relativity is hard. A fast interpretation of the equation is the one given by the physicist John Wheeler which said that matter tells space-time how to curve, and space-time tells matter how to move.

The first two terms in the left tell us how the space curves, the third term in the right which is often dismissed in Einstein field equations, is the cosmological constant. In cosmology it's very important since it's related to dark energy, some kind of mysterious energy that makes the universe's expansion accelerate. In fact the most important difference between the "newtonian" friedmann equation and the one derived from general relativity is that in the second one we get a term for the cosmological constant, so we would get the following equation:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}$$

For the purposes of this project we haven't talked about the accelerated expansion because of the cosmological constant, we will just set it equal to 0 and obviate it when we analyse our results for Hubble's constant.

4. PRACTICAL RESEARCH

4.1. Calculating Hubble’s constant

With the theoretical basis we already have we are ready to go a step further and calculate Hubble's constant while proving that the universe is indeed expanding. To do that we will obtain the spectrum of different galaxies and calculate their redshift. As we know when space expands the wavelength increases as a consequence of it.

On the other hand we will do photometry with type I-A supernovae to calculate their distances. The observation of type I-A supernovae will be done from the observatory of L’Astronomica de Sabadell. To make things easier, since to do spectroscopy very bright objects are needed, we will use the data from the Weizmann Interactive Supernova Data Repository (WISeREP) to know the supernovae spectra and redshift.
This section will be divided into two, in the first one the magnitude of a type I-A supernovae from our observations will be calculated, in the second section we will measure the redshift and recessional velocity from the supernovae from the WISeREP.

4.2. Photometry of type I-A supernovae

With the data provided by L’Astronomica de Sabadell and the observations realized in their observatory during their Cosmology campus we have different supernovae to calculate their distance from us and relate it to the observed redshift from the WISeREP spectra.

We will start by calculating the magnitude (brightness) of the observed formula, to do that we will get the darks, flats (they are pictures used to remove any noise from the picture, to do dark frames we just need to cover the camera, while to do flats we need to illuminate it uniformly) and the images of the objects.

After that, we will get our pictures with the noise removed. Then with the help of the charts from the *AAVSO International Database* we will search for stars around the supernova whose magnitudes are known to use them as reference and compare their brightness to the supernova to know its magnitude.

The procedure is as follows: Some fraction of the photons of the photography impact each pixel of the camera, which results in an electron being excited and going to a capacitor, which is then used to make the picture. The voltage caused by the charge of the displaced electrons is converted into a digital number called ADU which is just the gain of the pixel. We then compare the ADU of the supernova and the stars to get a magnitude, that we can then relate to the real magnitude, which we know, of the stars and calculate the magnitude of the supernova.
We first calculate a magnitude from the ADUs with the following equation:

$$m^i calculated = -2,5\, log(ADUs)$$

We then subtract the real magnitude by the calculated one:

$$C_i = m^i real - m^i calculated$$

We finally find an average value for the differences:

$$C = \frac{1}{N}\sum C_i$$

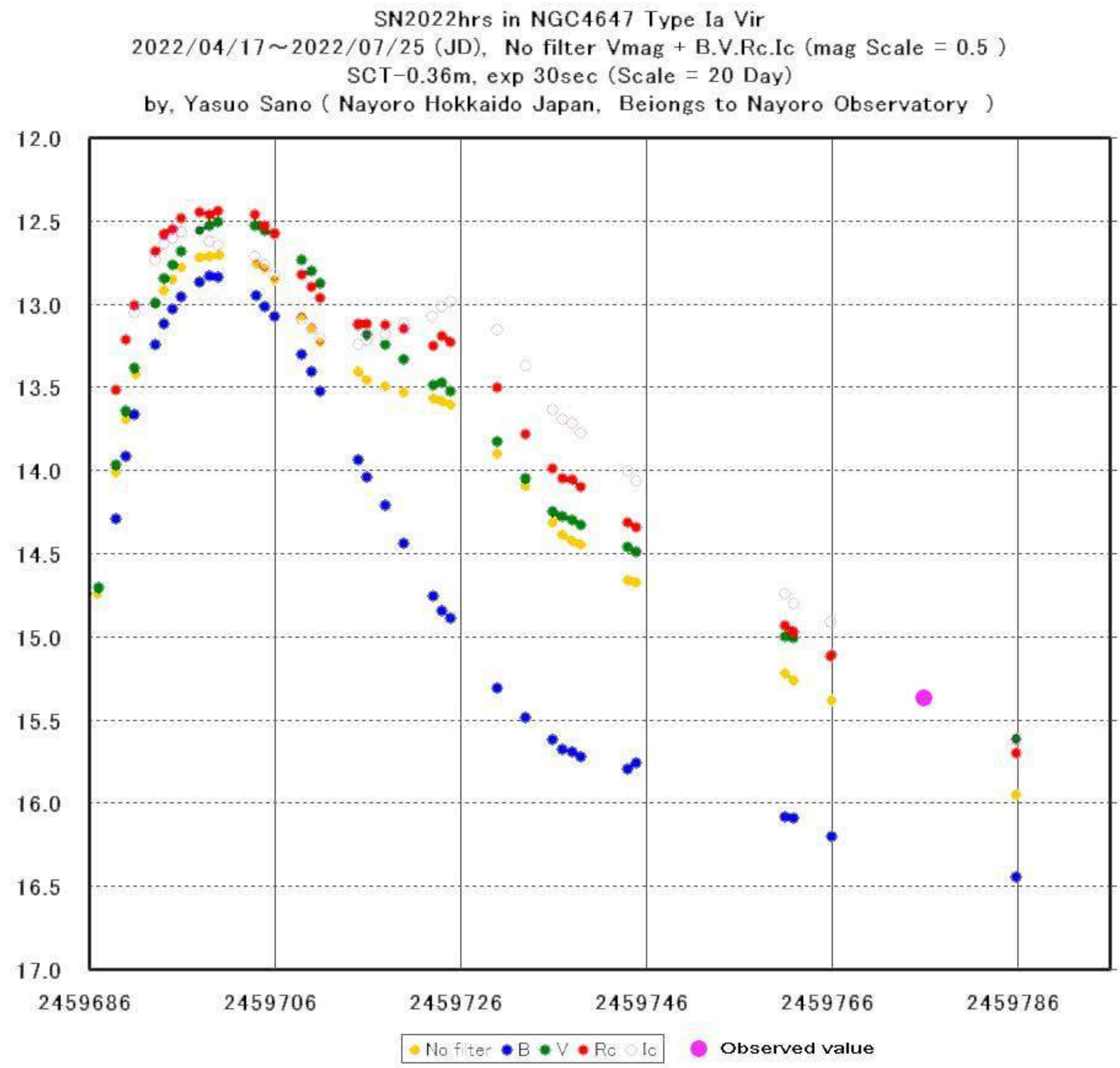
We now can calculate the real magnitude of the supernova from the image by using the first formula and adding C:

$$m^i calculated = -2,5\, log(ADUs) + C.$$

The first observed supernova is SN 2022 HRS, we have obtained a value of 15.4 in the V (visual) filter.

Photo of the supernova SN 2022hrs taken from Sabadell's observatory in july 15th 2022.

However we can't calculate the distance from this magnitude, since the supernova occured in spring and the value of the supernova has decreased since then, so we will see if our observations fit the curve made by other observations, if it does then we can take the peak value of the curve to use it to calculate hubble's constant. From the data published by Yasuo Sano from Nayoro observatory we have the following curve:



Graph showing how the brightness of the supernova has evolved. Yasuo Sano (Nayoro, Hokkaido,Japan)

Our picture was taken on July 15th 2022. In julian date it's 2459776.45833. Our observation seems to fit in the curve. Knowing the peak date and how it evolved we could try to find a mathematical model that fits in our point and we might find the actual peak value, but for the purposes of this project, the fact that our observations fit in the curve, allows us to take the peak value from Nayoro observatory as valid. In fact the best approach would have been to do multiple observations to see how the curve evolves and find the peak value by ourselves. For now we will just take 12.4 as the magnitude for our calculations.

We know that the absolute value of type I-A supernovae is -19.5. We have an equation that relates the apparent magnitude with the absolute one in function of the distance:

$$m = M - 5 + 5\, log_{10}(D)$$

Where m is the apparent magnitude, M the absolute one and D the distance. Rearranging we can get an expression for D:

$$D = 10^{\frac{(m-M)+5}{5}}$$

In our case, substituting the magnitudes we get a distance of 23988329,15 parsecs. A parsec is an astronomical unit of distance defined such that if you make a triangle with the sun, the earth and the point, and a right angle is formed in the sun the angle formed in the point is of one arcsecond.

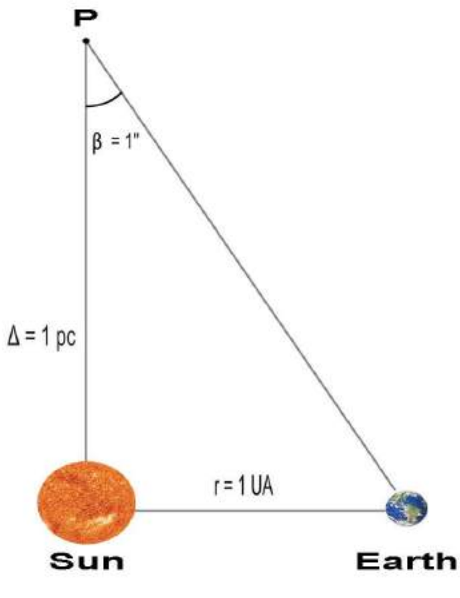


Image showing how a parsec is described from the distance between the earth and the sun

We can now convert our distance to megaparsecs which is the unit of distance used to calculate the Hubble's constant. We just have to divide our result by 1 000 000 to get:

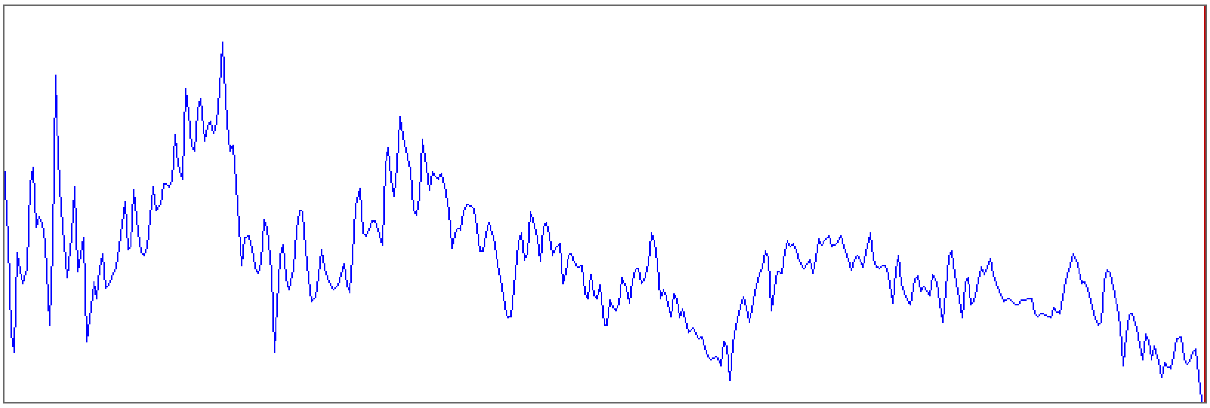
$$D = 23{,}98832915 \text{ Mpc}$$

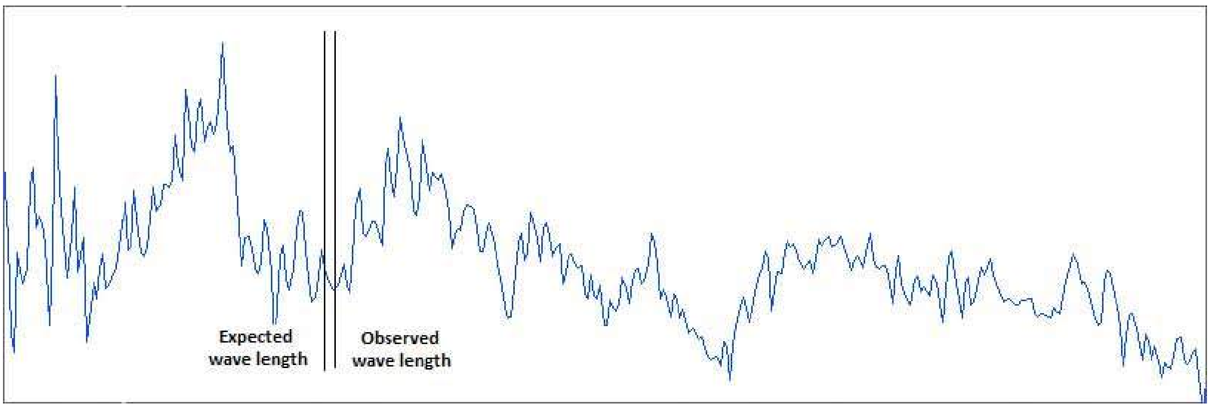
4.3. Calculation of the recessional velocity

Now we need to know at what velocity the supernova is going relative to the earth. From the WISeREP database we get the following spectrum observed by Claudio Balcon:

Spectrum of SN 2022hrs. Published in the WISeREP database. Claudio Balcon, April 16th 2022.

Or in a graph:





As we see the spectrum is shifted to the right (redshifted). In the second graph we can see the shift of the Hidrogenβ emission line. The hydrogen line should be observed at a wavelength of 4861.33 Å, but in our spectrum it's displaced to the right at 4884.18 Å. To calculate the redshift we will use the following formula:

$$z = \frac{\lambda observed - \lambda rest}{\lambda rest}$$

We get a redshift of z = 0.0047. To know at what velocity the supernova is going relative to us as consequence of the expansion of the universe we just need to use the following formula:

$$\sqrt{\frac{1+v/c}{1-v/c}} - 1$$

Which for low velocities where v >> c, that means where the recessional velocity is much lower than the speed of light, can be simplified as:

$$z = \frac{v}{c}$$

Or:

$$v = z \cdot c$$

So substituting by c by 300 000 km/s:

$$v = 0.0047 \cdot 300\,000\ km/s$$

There are no units for the value of z because it’s dimensionless, as we can see with the last equations. We get a recessional velocity of $v =$ 1410 km/s. From the definition of Hubble’s constant given previously:

$$H(t) = \frac{v}{D}$$

With our velocity and distances for the supernova SN 2020hrs we get:

$$H(t) = \frac{1410\ km/s}{23{,}99\ mpc} \approx 58.77\ km \cdot s^{-1} \cdot mpc^{-1}$$

We get a value for Hubble’s constant of 58.77 km/s! In the following section we will analyze the value of different cosmological parameters from our value for hubble’s constant.

5. DISCUSSION

ANALYZING OUR RESULTS

To start with we can calculate the value for the age of the universe from our value for Hubble's constant, it is relatively easy, we just need to convert our units.

We have that:

$$H(t) = \frac{1410\ km}{23,99\ mpc \cdot s}$$

Note that that we have units of distance both in the numerator and the denominator, that means that if we convert the megaparsecs to kilometers we will have units of "s^{-1}", hence "1/H" must have units of time, it is in fact the value of the age of the universe. The reason it is this way is due to the fact that Hubble's constant is defined as "$\frac{V}{D}$" which from the equation "$D = V \cdot T$" we see that it's just an expression for time. However this supposes a linear relationship between "D" and "V", but the fact is that the expansion of the universe has accelerated and decelerated within time, so the relationship isn't linear, however it turns out that at this point in time "D"and "V" approach to a linear relationship. So we have:

$$H(t) = \frac{23,99\ mpc \cdot s}{1410\ km} \cdot \frac{3,09{\cdot}10^{19}\ km}{1\ mpc} \cdot \frac{1\ year}{3,1536 \cdot 10^{7}\ s} \approx \text{16,67 billion years}$$

We have a value for the age of the universe!

Now we can use our value for the constant to calculate the value of the density of the universe supposing it is flat, which coincides with recent observations[1]. The expression for the critical density is:

$$\rho = \frac{3H^2(t)}{8\pi G}$$

We first need to convert our value for Hubble's constant to SI units:

$$H(t) = \frac{1410\ km}{23{,}99\ mpc \cdot s} \cdot \frac{1\ mpc}{3{,}09 \cdot 10^{19}\ km} \approx 1.90 \cdot 10^{-18} s^{-1}$$

Now substituting for "H" and "G" we get:

$$\rho = \frac{3 \cdot (1.90 \cdot 10^{-18} \cdot s^{-1})^2}{8\pi \cdot 6.6743 \cdot 10^{-11} m^3 \cdot kg^{-1} \cdot s^{-2}} \approx 6{,}47 \cdot 10^{-27}\ kg \cdot m^{-3}$$

Let's try to visualize our result. Let's suppose a grain of rice weighs 0.02 grams (We could have taken any other thing to compare it, such as a grain of sand, but it would be hard to get a value for its weight since every grain of sand is unique and has a different size and weight). Let's think we want to have a box with a grain of rice that has the same density as the universe. Let's see what volume the box would have. We just need to divide the weight of a grain of rice by the density we calculated:

$$0{,}02\ g \cdot \frac{1\ kg}{10^3\ g} \cdot \frac{1\ m^3}{6{,}47 \cdot 10^{-27}\ kg} \approx 1{,}29 \cdot 10^{23}\ m^3$$

Which in cubic kilometers is:

$$1{,}29 \cdot 10^{23}\ m^3 \cdot \frac{1\ km^3}{10^9 m} = 1{,}29 \cdot 10^{14}\ km^3$$

That is about twice the size of Neptune, or about 119 Earths!

That means that the density of the universe is about a grain of rice in 119 Earths!

1 - Ade, P. A., Aghanim, N., Armitage-Caplan, C., Arnaud, M., Ashdown, M., Atrio-Barandela, F., ... & Meinhold, P. R. (2014). Planck 2013 results. XVI. Cosmological parameters. *Astronomy & Astrophysics*, *571*, A16.

6. CONCLUSIONS (FR)

L'humanité a toujours été intéressée par l'univers est il a été facile de trouver des références à des succès astronomiques, même dans des sociétés marqués par la religion comme pendant la moyen age.

Le modèle astronomique actuel dépend du travail de beaucoup de scientifiques, certains dont je n'ai pas pu parler. Dans ces scientifiques on y trouve des femmes bien qu'elles aient été généralement ignorées dans le monde scientifique, elles ont eu un rôle important, en effet les premières valeurs de la constante de Hubble ont été calculées grâce aux travaux d'Enrietta Leavitt

On a prouvé avec nos observations et données réelles que notre univers est en expansion, et on a pu calculer sa taxe d'expansion et âge avec une précision relativement haute, étant donné qu'on a ignoré des choses comme l'énergie noire et on a utilisé qu'une seule supernova de type I-A. D'autre part on a utilisé des données d'observatoires avec une grande précision, ce qui a amené notre valeur pour la constante de Hubble à être proche de celle estimée. En conséquence, notre valeur pour la densité critique de l'univers est aussi assez proche de la valeur estimée. Il faudrait observer plus supernovas pour pouvoir avoir une valeur plus exacte, mais on peut conclur que nous avons réussi à obtenir des valeurs pertinentes pour calculer l'expansion de l'univers avec un petit budget

On peut conclure que malgré l'accélération initiale et la force de la gravité, l'expansion de l'univers a été plus ou moins linéaire, donc en supposant que notre univers était linéaire nous avons obtenu des résultats proches du réel. Nous avons aussi vu que les estimations de la densité de l'univers de différents projets, correspondent au cas d'un univers plat.

On a pu voir comment l'expansion de l'univers peut être expliqué sans avoir besoin de la relativité générale, fait qui peut approcher la théorie du big bang à des gens sans formation académique. En fait l'unique différence relevante qu'il y a entre l'équation de friedman newtonienne et l'équation dérivée de la relativité générale c’est la constante cosmologique. On peut donc conclure que l'expansion de l'univers peut être étudiée avec les notions enseignées au baccalauréat. Que nous ayons réussi à approcher les principaux concepts de la cosmologie à un public sans grande formation doit être décidé par le lecteur.

7. BIBLIOGRAPHY

Books:

- MALVILLE, J. M. K. (2008) - *A guide to prehistoric astronomy in the Southwest* - Ed. Johnson Books
- *Dietari General de Catalunya .* Volume I. (1411-1539).
- For further information about astronomical references in Dietari de la Generalitat see: JOSÉ, M., & MARCO, F. J. Astronomical references in the first volume of the Dietaris de la Generalitat de Catalunya (1411-1539) and the the Dietari o llibre de Jornades de Jaume Safont (1411-1484).
- GAMOW, G. (1970) - *My Worldline* - Ed.Viking Press.
- LEMAÎTRE, G. (1931) - The Beginning of the World from the Point of View of Quantum Theory - Ed.*Nature*.
- SMOOT, G., & DAVIDSON, K. (2007) - In *Wrinkles in time: Witness to the birth of the universe* - essay - Harper Perennial.
- SUSSKIND, L. & HARABOVSKY, G. (2013) - Classical Mechanics, The Theoretical Minimum - Ed. Penguin Science.
- SUSSKIND, L. & FRIEDMAN, A. (2017) - Special Relativity and Classical Field Theory, The Theoretical Minimum - Ed. Penguin Science.
- EINSTEIN, A. (1916) - “Sobre la teoría de la relatividad especial y general” - Alianza editorial (2017).
- GREENE, B. (1999) - “El universo elegante” - Ed. Critica.
- ROVELLI, C. (2017) - “L’ordre del temps” - Ed. Anagrama.
- PENROSE, R. (2010) - “Ciclos del Tiempo” - Ed. Penguin Random House.
- COX, B. & FORSHAW (2016) - Universal a journey through the cosmos - Ed. Penguin Random House.
- GREENE, B (2004) - “El tejido del cosmos” - Ed. Planeta
- PENROSE, R (2004) - “El camino a la realidad”

Webs:

- *Edwin Hubble*. Biografias y Vidas. (n.d.). Retrieved September 23, 2022, from https://www.biografiasyvidas.com/biografia/h/hubble.htm
- Archive, B. / B., Mario García Bartual 24/04/2020 08:00 Actualizado a 24/04/2020 12:27, Bartual, M. G., público, D., & BY-SA-2.0, A. D. / C. C. (2020, April 24). *Edwin Hubble: ¿Qué aportó a la ciencia?* La Vanguardia. Retrieved September 29, 2022, from https://www.lavanguardia.com/historiayvida/historia-contemporanea/20200424/48676604818/edwin-hubble-planetas-astronomia-via-lactea.html
- *Hubble y la Expansión del Universo*. Edwin Hubble y la expansión del Universo. (n.d.). Retrieved September 29, 2022, from https://www.astromia.com/biografias/hubble.htm
- *Edwin Hubble y la Expansión del Universo - astrosabadell.org*. (n.d.). Retrieved September 29, 2022, from https://astrosabadell.org/pdf/es/bio/homes/Hubble_es.pdf
- *George Gamow*. Biografias y Vidas. (n.d.). Retrieved September 29, 2022, from https://www.biografiasyvidas.com/biografia/g/gamow.htm
- *George Gamow, El Padre de la Teoría del Big Bang - astrosabadell.org*. (n.d.). Retrieved September 29, 2022, from https://astrosabadell.org/pdf/es/bio/homes/Gamow_es.pdf
- Montoya, P. L. (2022, June 3). *George Gamow*. Historia y biografía de. Retrieved September 29, 2022, from https://historia-biografia.com/george-gamow/

Lectures:

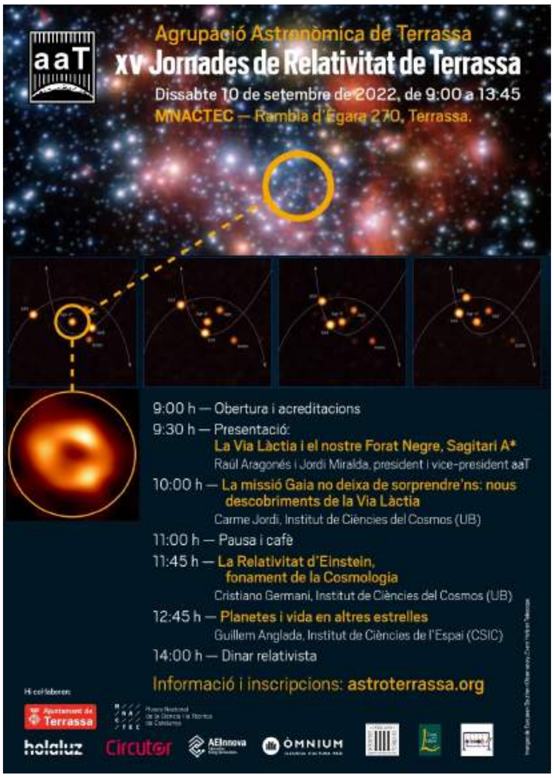
- SUSSKIND, L. (2009) Leonard Susskind's lectures on Cosmology. Stanford University

Campus:

"*Associació Astronòmica de Sabadell*"

Conferences:

- Cristiano Germani (UB) - "La Relativitat d'Einstein, fonament de la Cosmologia" - Agrupació Astronòmica de Terrassa - 10/09/2022.
- Carme Jordi (UB) - "La missió Gaia no deixa de sorprendre'ns: nous descobriments de la Via Làctia" - Agrupació Astronòmica de Terrassa - 10/09/2022.



- Tomas Andrade (UB) "El sueño va sobre el tiempo: agujeros negros y ondas gravitacionales" - Tardes de Ciencia, AstroBanyoles - 09/10/2021.
- Alicia Sintes - "Les ones gravitacionals: les noves missatgeres de l'Univers" - Tardes de Ciencia, AstroBanyoles - 08/05/2021.
- Carme Jordi (UB) - "GAIA, mil milions d'estrelles" - Tardes de Ciencia, AstroBanyoles - 11/01/2020.

https://www.astrobanyoles.org/codi/tardesdeciencia.php

Data of supernova obtained:

- Publications of the Astronomical Society of the Pacific, Volume 124, Issue 917, pp. 668 (2012) - Kafka, S., 2021, Observations from the AAVSO International Database, https://www.aavso.org.
- *Home*. WISeREP. (n.d.). Retrieved August 30, 2022, from https://wiserep.org/
- Publications of the Astronomical Society of the Pacific, Volume 124, Issue 917, pp. 668 (2012).

1. APPENDIX:

Equivalence between mass and energy

The equivalence between mass and energy will be derived in the appendix given that it involves a higher knowledge of math and to do it we need to use the Euler-Lagrange equations and the principle of least action which we won't prove, but we will just demonstrate its equivalence to the newtonian laws of motion.

To do that we just need the 4D invariant metric we derived. Remember that the 4D metric gave us a value that was invariant and it was related to the time for a person in the origin of a coordinate frame of reference.

Remember that in classical mechanics we define velocity as the variation of position over time, since it's an absolute thing. But in special relativity time is no longer an absolute thing, how can we define the velocity over space-time? We can use the proper time since it's absolute.

With that we can define the four components of a 4D velocity:

$$U^{\mu} = \frac{dX^{\mu}}{d\tau}$$

Note that we use a U instead of V to avoid confusing it with the classical velocity, and we use greek letters instead of latin ones to indicate we are summing over space time coordinates. The components of the 4-velocity are:

$$U^{0} = \frac{dX^{0}}{d\tau} = \frac{dt}{d\tau}$$

$$U^{1} = \frac{dX^{1}}{d\tau} = \frac{dx}{d\tau}$$

$$U^{2} = \frac{dX^{2}}{d\tau} = \frac{dy}{d\tau}$$

$$U^{3} = \frac{dX^{3}}{d\tau} = \frac{dz}{d\tau}$$

How can we relate our 4 velocity with the ordinary velocity? Let's use the chain rule, we will start with the time component and then analyze the spatial components.
With the chain rule we can write:

$$U^{\mu} = \frac{dX^{\mu}}{d\tau} = \frac{dX^{\mu}}{dt}\frac{dt}{d\tau}$$

Now we can get an expression for $\frac{dt}{d\tau}$ by substituting $d\tau$ by $\sqrt{dt^2 - dx^2}$, to make things easier we will just find $\frac{d\tau}{dt}$:

$$\frac{d\tau}{dt} = \frac{\sqrt{dt^2 - dx^2}}{\sqrt{dt^2}} = \sqrt{1 - \frac{dx^2}{dt^2}} = \sqrt{1 - v^2}$$

Now we can substitute this expression to find the components of the 4-D velocity. We are going to use 0 for the time component and i for the space components:

$$U^0 = \frac{dX^0}{dt}\frac{dt}{d\tau} = \frac{dt}{dt}\frac{1}{d\tau/dt} = \frac{1}{\sqrt{1-v^2}}$$

$$U^i = \frac{dX^i}{dt}\frac{dt}{d\tau} = V^i\frac{1}{d\tau/dt} = \frac{V^i}{\sqrt{1-v^2}}$$

And we find a very interesting property, that is that as the four components are invariant, their velocities are invariant:

$$(U^0)^2 - (U^1)^2 - (U^2)^2 - (U^3)^2 = k$$

In fact we can show that k is equal to 1:

$$k = (\frac{1}{\sqrt{1-v^2}})^2 - (\frac{v}{\sqrt{1-v^2}})^2$$

$$k = \frac{1}{1-v^2} - \frac{v^2}{1-v^2}$$

$$k = \frac{1-v^2}{1-v^2} = 1$$

That shows us a beautiful relationship between space and time, the slower you move along the space axis, the faster you move along time, and the faster you move along the space axis, the slower you move along time. That means that in terms of the 4-D invariant velocity we are moving at the same speed as light in space-time.

To continue we have to review a mathematical approximation tool to prove that in the non relativistic limit, that means when objects move at slow speeds compared to light, our equations give the classical ones.

Suppose we have an expression similar to $(1 + a)^3$, we can expand the expression writing:

$$(1 + a)^3 = 1 + 3a + 3a^2 + a^3$$

But notice that for small numbers in a, such as the case of the square of the ratio between the ordinary velocity of a particle and the speed of light, the higher expressions of a can be ignored. Similarly, when we have $\sqrt{1 - v^2}$ and $\frac{1}{\sqrt{1-v^2}}$ we can ignore the higher components of v, so we have the following approximations:

$$\sqrt{1 - v^2} = (1 - v^2)^{1/2} = 1 - \frac{1}{2}v^2$$

$$\frac{1}{\sqrt{1-v^2}}(1 - v^2)^{-1/2} = 1 + \frac{1}{2}v^2$$

To proceed we have to review the principle of least action. Suppose we know that a system starts at some point a and ends up at b and we know the energies acting over it. We can define a value called action that depends on the lagrangian (which is a function of the energies acting over the system) defined as:

$$Action = \int_a^b L\,dt$$

Where:

$$L = Kinetic\ Energy - Potential\ Energy$$

The action tells us how a physical system changes over time. And we know that the energy of a system tends to change as little as possible so we know that the change of the action of a particle must be 0.

$$\delta Action = \delta \int_a^b L\,dt = 0$$

The mathematical solution of this equation is:

$$\frac{d}{dt}\frac{\partial L}{\partial v} - \frac{\partial L}{\partial X} = 0$$

The explanation is not totally accurate and the justification we have done is not quite correct, but the main point is that the principle of least action tells us how a system evolves by just knowing two points in it. In fact the solution of the equation is just an equivalent of Newton's second law in terms of energy, let's see how. Let's suppose we have a system whose lagrangian is defined by $L = T - V$ where T stands for the kinetic energy of a particle and V stands for the potential energy, which is given by some force. Then it's lagrangian will be:

$$L = \frac{1}{2}mv^2 - V(x)$$

Let's use the solution of minimizing the action we have to get the equation of motion of a system. Let's calculate the term $\frac{\partial L}{\partial v}$:

$$\frac{\partial L}{\partial v} = \frac{\partial L}{\partial v}(\frac{1}{2}mv^2 - V(x)) = mv$$

The time derivative of the expression we have is just mass times the acceleration, so we have:

$$\frac{d}{dt}\frac{\partial L}{\partial v} = ma$$

Now we can calculate the other term:

$$\frac{\partial L}{\partial X} =- \frac{\partial V}{\partial X}$$

From the definition of the potential energy:

$$V =- \int F dx$$

Derivating in both sides we have:

$$\frac{\partial V}{\partial X} = \frac{\partial}{\partial X}(- \int F dx)$$

The integral and derivative cancel so we get:

$$\frac{\partial V}{\partial X} =- F$$

$$F =- \frac{\partial V}{\partial X}$$

So the equation for our particle is:

$$\frac{d}{dt}\frac{\partial L}{\partial v} - \frac{\partial L}{\partial X} = ma - F = 0$$

Or:

$$F = ma$$

In other words, we have proved that the principle of least action is just an expression in terms of energy of Newton's second law of motion.

Finally we have to see the expression for the total energy of a system, called Hamiltonian, from the Lagrangian the expression is given by:

$$H = \sum_i \dot{X}^i P^i - L$$

Where $\dot{X}$ is the velocity. P the momentum and L the lagrangian, the i stands for the three components of position. If the expression is correct we should get the sum of the kinetic and potential energies, let's check it:

$$H = v(mv) - (\frac{1}{2}mv^2 - V(x))$$

$$H = mv^2 - \frac{1}{2}mv^2 + V(x)$$

$$H = \frac{1}{2}mv^2 + V(x) = T + V$$

Now that we know the Lagrangian and the Hamiltonian and we have proved them to work for a non relativistic particle, let's now find the relativistic equations of motion for a simple particle with no forces acting on it.

Let's start with our relativistic lagrangian, we know that our equations of motion must be valid in all reference frames, but in relativity time is no longer something absolute, that's why our lagrangian cannot depend on time. But we know a value that is invariant, that's τ. So our action will be an integral over τ instead of t:

$$Action = k\int_a^b d\tau$$

We need equations that work in the non relativistic limit, and we know that they depend on the particle's mass, so the action must be proportional to the mass of the particle. In fact, if our equation is valid, it should match the classical equation for kinetic energy at a non relativistic velocity. If we worked the equations with a positive m (mass) we would find that the only way to match the expression with the classical kinetic energy is to add a negative sign, otherwise we would get a negative kinetic energy.

So our action will be:

$$Action = -m\int_a^b d\tau$$

Finally there's a way to plug time in our equation and make it still invariant, we just have to use the equation we derived earlier:

$$\frac{d\tau}{dt} = \sqrt{1 - v^2}$$

$$d\tau = \sqrt{1 - v^2}dt$$

So our action is:

$$Action = -m\int_a^b \sqrt{1 - v^2}dt$$

Now since the expression for the action is:

$$Action = \int_a^b L\, d\tau$$

We can deduce the expression for our relativistic lagrangian:

$$L = -m\sqrt{1 - v^2}$$

Or restoring units (that means multiplying by c since it's an invariant value to get units of energy):

$$L = - mc^2\sqrt{1 - \frac{v^2}{c^2}}$$

Now we can use our approximation for the square root:

$$\sqrt{1 - \frac{v^2}{c^2}} = 1 - \frac{1}{2}\frac{v^2}{c^2}$$

So for slow velocities, the non relativistic lagrangian will be:

$$L = - mc^2(1 - \frac{1}{2}\frac{v^2}{c^2})$$

$$L = - mc^2 + mc^2(\frac{1}{2}\frac{v^2}{c^2})$$

$$L = \frac{1}{2}mv^2 - mc^2$$

The second term is our good old kinetic energy! What about the first term? We know that it doesn't affect the motion of the particle since it's a constant and it vanishes when we take the derivatives of the lagrangian, but what meaning does it have? That's what we are going to see.

To find the total energy of the relativistic particle we need its Hamiltonian. But first we have to get an expression for its momentum. In fact the momentum can be derived from the Lagrangian:

$$P^x = \frac{\partial L}{\partial v}$$

The reader can try to use this definition in the classical lagrangian of the particle, and will see that it gives $P = mv$.

Let’s now apply it to our relativistic lagrangian:

$$P = \frac{\partial L}{\partial v} = \frac{\partial}{\partial v}\left(- mc^2 \sqrt{1 - \frac{v^2}{c^2}}\right)$$

$$P = m\frac{v}{\sqrt{1-\frac{v^2}{c^2}}}$$

Or taking into account the three components of the velocity:

$$P^i = m\frac{V^i}{\sqrt{1-\frac{v^2}{c^2}}}$$

Notice that this is just the mass times the spatial components of the 4-D velocity, so we can write:

$$P^i = mU^i.$$

Now we have enough to calculate our relativistic hamiltonian, or the total energy of our relativistic system. We know that:

$$H = \sum_i V^i P^i - L$$

Substituting:

$$H = \sum_i V^i mU^i - \left(- m\sqrt{1 - v^2}\right)$$

$$H = \sum_i V^i m \frac{V^i}{\sqrt{1-v^2}} + m\sqrt{1 - v^2}$$

$$H = \sum_i \frac{m(V^i)^2}{\sqrt{1-v^2}} + m\sqrt{1 - v^2}$$

$$H = \frac{mv^2}{\sqrt{1-v^2}} + \frac{m(1-v^2)}{\sqrt{1-v^2}}$$

$$H = \frac{mv^2+m-mv^2}{\sqrt{1-v^2}}$$

$$H = \frac{m}{\sqrt{1-v^2}} = E$$

Notice that $\frac{1}{\sqrt{1-v^2}}$ is just the time component of the 4-D velocity, so we can write:

$$E = P^0 = mU^0$$

The time component momentum of our particle is just its total energy!

Now restoring our units:

$$E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

For small values of v/c we get:

$$E = mc^2 + \frac{1}{2}mv^2$$

So the total energy of a particle is its kinetic energy plus a constant called its rest energy. So when the velocity of an object is zero, it’s energy is given by:

$$E = mc^2$$

What about the energy of a massless particle? Let's remember the expression of the 4-D velocities:

$$(U^0)^2 - (U^1)^2 - (U^2)^2 - (U^3)^2 = 1$$

We can now multiply both sides by m^2:

$$m^2(U^0)^2 - m^2(U^1)^2 - m^2(U^2)^2 - m^2(U^3)^2 = m^2$$

Notice that mU^μ is just the relativistic momentum, so we can write:

$$(P^0)^2 - (P^1)^2 - (P^2)^2 - (P^3)^2 = m^2$$

The first term P^0 is just the energy, and the others form the 3-D momentum, so it boils down to:

$$E^2 - P^2 = m^2$$

$$E = \sqrt{m^2 + P^2}$$

And multiplying by c to restore our units:

$$E = \sqrt{m^2c^4 + P^2c^2}$$

In the case of an object at rest we get back:

$$E = \sqrt{m^2c^4 + 0}$$

$$E = mc^2$$

For the case of a massless particle like a photon:

$$E = \sqrt{0 + P^2c^2}$$

$$E = c|P|$$

That’s the expression for the energy of a Photon! In fact it also holds approximately for neutrinos.

Note that we use the magnitude to ensure that E is always a real number.

We have just made some simple assumptions and we have ended up with an expression that tells us that mass and energy are equivalent, and another that tells us the energy and momentum of a Photon! That’s the beauty of the 4-D relativistic metric.